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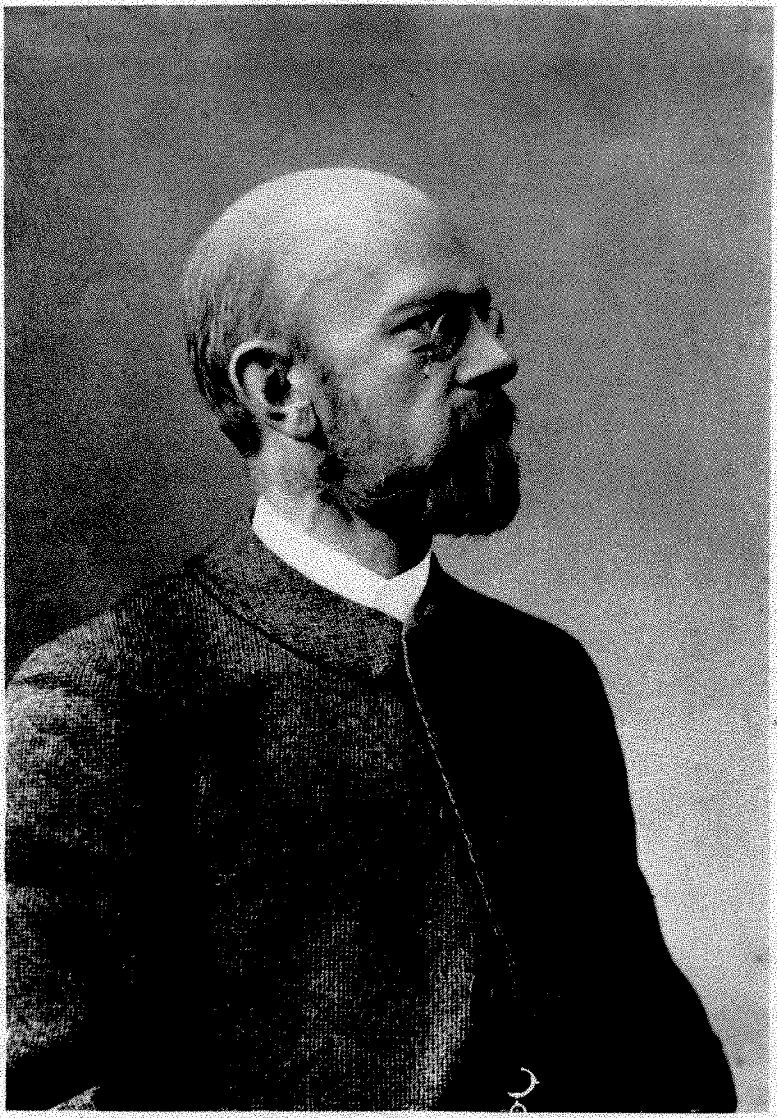

David Hilbert

p14729

The Group which Contain less than Fifteen Operators of Order Two.

By G. A. Miller.

The groups which contain exactly thirteen operators of order two have recently been considered.* These are of somewhat special interest since they do not include any group whose order is a power of 2. For every smaller possible number (n) of operators of order 2 there is at least one group whose order is of the form 2^m. It is well known that every group (G) which contains at least one operator of order 2 contains an odd number of such operators. Hence we may assume that n represents successively all the odd numbers from 1 to 11.

When the operators of order two in G do not generate the entire group, they generate a characteristic subgroup H. The object of the present paper is a complete determination of H for every value of n within the given limits, and a determination of some properties of G which result directly from H. The determination of all the possible G's for any value of n is a much more difficult problem, which involves the determination of all groups of odd order. It is believed that a knowledge of all the possible H's for small values of n will be useful in the study of many other problems.

When $n = 1$, H is the cyclic group of order 2. When $n = 3$, two of these operators generate a dihedral rotation group which includes the third. Hence, H is either the symmetric group of order 6 or the four-group, as the operators of order two are non-commutative or commutative. Each of these two groups has the symmetric group of order 6 for its group of isomorphisms. They are so well known that it seems unnecessary to specify other properties.

Since $5 \equiv 1 \pmod 4$ there are exactly two groups whose order is of the form 2^m and which contain only 5 operators of order 2;† viz., the octic group and the group of order 16 which involves operators of order 8 that are transformed into their third powers. The former of these is generated by its operators of order 2, while these operators generate the octic subgroup of the latter. Hence there is only one group of order 2^m which contains only five operators of order 2 and is generated by these operators, viz., the octic group, or the group of the square. It remains to determine the possible H whose order is not of the form 2^m.

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* Bulletin of the American Mathematical Society, Vol. XII, 1906, p. 389.

† Transactions of the American Mathematical Society, Vol. 6, 1905, p. 62.

Two of the 5 operators of order 2 generate a dihedral rotation group whose order cannot exceed 10. If it is 10 any pair of operators of order 2 generate it. If it were 6 the two remaining operators of order 2 could not transform this subgroup into itself since a group of order 12 cannot involve exactly 5 operators of order 2. Nor could one of these operators transform this subgroup into a different subgroup since the latter would have at least two operators of order 2 which would not be in the former subgroup. The product of each pair of operators of order 2 could not be of order 2 since $5 \equiv 1 \bmod 2^\alpha$. Hence, *the octic group and the dihedral rotation group of order* 10 *are the only two groups which contain exactly* 5 *operators of order two and are generated by these operators.*

§ 1. *General Theorems on Dihedral Rotation Groups.*

The following general theorems are of interest in themselves and will greatly simplify the consideration of the remaining cases.

THEOREM I. *If an operator* (t) *of order two transforms into itself a dihedral rotation group* (D) *whose order* (d) *is not divisible by* 4, *the group* $[D, t]$ *generated by* D *and* t *is the direct product of two dihedral rotation groups.**

To prove this theorem we shall first consider the special case when $d = 2p^\alpha$, where p is an odd prime number. Let C represent the cyclic subgroup of order p^α contained in D. Since the group of isomorphisms of C is cyclic it follows that t transforms C according to an operator in the group of cogredient isomorphisms of D. That is, t transforms either each operator of C into its inverse or it is commutative with each of these operators. In the latter case it is also commutative with the operators of order 2 in D since it could not transform one of these into itself multiplied by an operator of odd order with which it is commutative. In the former case it is commutative with an operator of order 2 in D since it cannot have d conjugates under D. Hence t is always commutative with at least one operator of order 2 in D. As the product of these two operators is again of order 2 and could be used for t, it follows that $[D, t]$ *is the direct product of* D *and a group of order* 2 *whenever* $d = 2p^a$. This direct product is the dihedral rotation group of order $4p^a$.

---

*One of these dihedral rotation groups may be of order 2. In this case $[D, t]$ is the direct product of D and a group of order 2.

When t is commutative with any operator, besides the identity, in a Sylow subgroup of C it is commutative with all the operators of this Sylow subgroup. If t is non-commutative with any such operator it transforms all the operators of this Sylow subgroup into their inverses. In the latter case, t is commutative with an operator of order 2 in a dihedral rotation group containing this Sylow subgroup and being contained in D, since the number of such dihedral rotation groups is odd. In the former case, t is commutative with all the operators of such a dihedral rotation group for the same reason. Hence we have as before that t is commutative with at least one operator of order 2 contained in D.

The Sylow subgroups of C may be divided into two sets; the first being composed of those whose operators are commutative with t while the second includes those whose generators are transformed into their inverses by t. The direct product of the two groups generated by these two sets is C, and D may be constructed by dimidiating the two dihedral rotation groups which involve respectively these two factor groups of C. The number of operators of order 2 in D with which t is commutative is therefore equal to the order of the former of the two given factor groups of C, and *the number of operators of order* 2 *in* $[D, t]$ *which are not also in* D *is equal to the sum of the orders of these two factor groups*. Moreover, it is clear that t may be replaced by some operator of order 2 in the second of the two groups whose dimidiation gives rise to D. In other words, $[D, t]$ is the direct product of these two groups. This proves the theorem in question.

For our present purpose the possible number of operators of order 2 in $[D, t]$ is most important. From what precedes it follows that this number is $d/2$ plus the sum of any two relatively prime numbers whose product is $d/2$. In particular, if one of these two numbers is $d/2$ the other is unity. This special case leads to the maximum number of operators of order 2 in $[D, t]$ and is the only possible case when $d/2$ is a power of a prime, as was observed above. The main result may be expressed as follows: *the number of operators of order* 2 *in* $[D, t]$ *is* $d/2 + f_1 + f_2$, *where* f_1, f_2 *are relatively prime and* $f_1 f_2 = d/2$. The operator t may be so chosen that f_1, f_2 have any values satisfying the given conditions. If $f_1 = d/2$ it must be assumed that $f_2 = 1$.

THEOREM II. *If the order* (d) *of a dihedral rotation group* (D) *is divisible by* 4, *there is an operator* (t) *of order* 2 *such that* $t^{-1}Dt = D$ *and* $[D, t]$ *contains exactly* $d/2 + 3$ *operators of order* 2.

To obtain this group t must be chosen so as to be commutative with each operator of the cyclic subgroup of order $d/2$ contained in d and to transform the remaining operators into themselves multiplied by the operator of order 2 in this cyclic subgroup. The operators of $[D, t]$ which are not in D nor in the direct product of the cyclic subgroup of order $d/2$ contained in D and t are all of order 4 and have a common square. This is the only possible case in which a dihedral rotation group, extended by means of an operator of order 2, gives rise to a group which contains only two more operators of order 2 than the original group.

THEOREM III. *The group of isomorphisms of the dihedral rotation group of order* d *is the holomorph of the cyclic group of order* $d/2$, *except when* $d = 4$.*

When d is not divisible by 4 this theorem requires no proof since the group of cogredient isomorphisms of the holomorph of the cyclic group of order $d/2$ is the group of isomorphisms of the subgroup which is the dihedral rotation group (D) of order d. In general, the non-invariant operators of order 2 in D may be transformed according to the operators of the cyclic group of order $d/2$, without affecting the operators of the cyclic subgroup of this order. Hence the group of isomorphisms of D contains an invariant cyclic subgroup of order $d/2$. It can be represented as a substitution group on $d/2$ letters and includes operators which transform this cyclic subgroup in every possible manner. This completes the proof since D is invariant in the holomorph of the cyclic group of order $d/2$.

THEOREM IV. *If an operator* t *of order* 2 *transforms a dihedral rotation group* (D) *into itself, and is commutative with all the operators of a Sylow subgroup of order* 2^m *contained in* D, *then* $[D, t]$ *is the direct product of two dihedral rotation groups.*

Since t is commutative with a non-invariant operator t_1 of order 2 contained in D, it transforms into itself some dihedral rotation group involving any Sylow subgroup of the cyclic group of order $d/2$ contained in D. The product of these Sylow subgroups whose generators are transformed into their inverses by t together with t generates a dihedral rotation group which is invariant under $[D, t]$.

---

* The group of Theorem II. corresponds to the invariant operator of order 2 in this holomorph. Jordan, Traité des substitutions, 1870, p. 60.

The product of the remainder of these Sylow subgroups and tt_1 generate a second dihedral rotation group. As all the operators of one of these groups are commutative with each operator of the other it is clear that $[D, t]$ is their direct product. It may be observed that Theorem I can be directly derived from this theorem. On account of its importance it seemed desirable to prove it separately.

Suppose that the order of D is 2^m. If t transforms into itself each operator of the cyclic group (C) of order 2^{m-1} contained in D then $[D, t]$ is either the direct product of D and t, or it is the group of Theorem II. In the former case it contains $2^m + 3$ operators of order 2, while there are $2^{m-1} + 3$ such operators in the latter case. If t transforms each operator of C into its inverse, $[D, t]$ is either the given direct product or the dihedral rotation group of order 2^{m+1}. The latter contains $2^m + 1$ operators of order 2. The only other case which requires consideration is when t transforms every operator of C into its $2^{m-2} + 1$ power, $m > 3$. There are two conjugate operators of order 2 in the group of isomorphisms of D which have this property. If t represents one of these, $[D, t]$ contains $2^{m-1} + 2^{m-2} + 3$ operators of order 2. Hence the following:

THEOREM V. *There are exactly four groups of order* 2^{m+1} *which result by extending the dihedral rotation group of order* 2^m, $m > 3$ *by means of operators of order* 2. *These contain respectively* $2^{m-1} + 3$, $2^{m-1} + 2^{m-2} + 3$, $2^m + 1$, *and* $2^m + 3$ *operators of order* 2.

The second of these four groups does not exist as a distinct group when $m < 4$, when $m = 2$ the only possible groups are the octic and the abelian of type (1, 1, 1). If D represents any dihedral rotation group at least one of its Sylow subgroups is transformed into itself by t, whenever $t^{-1} D t = D$ and $t^2 = 1$. Hence the number of operators of order 2 in $[D, t]$ which are not also in D cannot be less than the difference between $2^{m-1} + 1$ and one of the four numbers given in Theorem V.

Suppose that G contains two dihedral rotation groups of order $2p$, p being a prime, and that these are conjugate under an operator (t) of order 2 contained in G. We proceed to prove that G contains at least $2p + 3$ operators of order 2. As this has been proved above for the case when $p = 2$ we shall assume that p exceeds 2 in what follows. To prove the theorem in question it is only necessary to prove that G cannot contain exactly $2p+1$ operators of order 2. We proceed to prove this fact.

Let s_1, s_2 represent two operators of order 2 which generate one of the two conjugate dihedral rotation groups of G and let D_1 represent this group. Since G contains only $p+1$ operators of order 2 besides those of D, it follows that $s_1 s_2$ transforms at least one of these (s_3) into itself. Hence s_3 transforms into itself the cyclic subgroup of order p contained in D_1. It cannot transform D_1 into itself as G contains an operator of order 2 which transforms D_1 into another group (D_2). Hence we may suppose that $s_3^{-1} D_1 s_3 = D_2$ and that D_1 and D_2 have p common operators, and therefore involve $2p$ distinct operators of order 2. As $s_3^{-1} s_1 s_3 = s_4$ is different from s_3 and is also commutative with the operators of order p contained in D_1, it has been proved that G cannot contain exactly $2p+1$ operators of order 2. Hence there results

THEOREM VI. *If a group contains two dihedral rotation groups whose order is twice a prime number* (p) *and if these two groups are conjugate under one of its operators of order two, then the group contains at least* $2p+3$ *operators of order* 2.

§ 2. *Groups which Contain exactly seven Operators of Order Two.*

When $n=7$ the dihedral rotation group generated by two of these operators (s_1, s_2) has one of the following numbers for its order: 4, 6, 8, 10, 12, 14. In the last two cases the group generated by s_1, s_2 includes the other five operators of order 2. Hence it remains to consider the cases when the order of $s_1 s_2$ does not exceed 5. From Theorems I and VI it follows directly that the order of $s_1 s_2$ cannot be 5.

If $s_1 s_2$ is of order 4 the octic group is generated by s_1, s_2. One of the remaining two operators of order 2 transforms this into itself as two octic groups have to contain at least 7 distinct operators of order 2. Hence the seven operators of order 2 generate the group of order 16 which involves exactly three octic groups. If $s_1 s_2$ were of order 3 an operator of order 2 which is not in $[s_1, s_2]$ would transform this group into itself according to Theorem VI. Hence the seven operators of order 2 would generate the dihedral rotation group of order 12 mentioned above. Finally, if the product of every two of these seven operators is of order 2, they generate the abelian group of order 8 and of type (1, 1, 1). Combining these results we have the following theorem: *If a group contains exactly seven operators of order* 2 *and is generated by these operators its order is* 8, 12, 14 or 16. In each case there is only one group. The first is abelian, the second and third are dihedral rotation groups, while the fourth is the only non-abelian group of order 16 which contains just three pairs of conjugates of order two. Every group which contains exactly 7 operators of order 2 must therefore contain one of these four groups as a characteristic subgroup.

§ 3. *Groups which Contain exactly nine Operators of Order Two.*

Since $9 \equiv 1 \bmod 4$ the dihedral rotation group of order 16 is the only group of order 2^m which contains exactly 9 operators of order 2 and is generated by these operators, and there is only one other group whose order is of the form 2^m and which contains exactly 9 operators of order 2. Before proceeding to the consideration of the groups whose orders are not of the form 2^m it seems desirable to prove a theorem which will be frequently employed in what follows.

THEOREM. *If a Sylow subgroup of any group* (G) *contains more than one subgroup of prime order* (p) *but does not contain all the subgroups of this order contained in* G, *then there are at least* p^2 *subgroups of order* p *in* G *besides those in a given Sylow subgroup.*

Before giving a proof of this theorem we shall give one illustration. The symmetric group of order 24 contains a Sylow subgroup of order 8 which contains five operators of order 2, but does not include all of the operators of this order in the symmetric group. The given theorem says that the number of additional operators of this order is at least 4. This gives the exact number of operators of order 2 in the symmetric group of order 24, viz. 9.

To prove the theorem we let S_1, S_2 represent two Sylow subgroups of G such that they have the largest possible number of common operators of order p without having all the operators of this order in common. These common operators generate a common subgroup which is transformed into itself by some other operator of order p in S_1. This operator transforms S_2 into p conjugates such that each has at least p subgroups of order p which are not in any of the others not contained in S_1. This proves the theorem.

From this theorem it follows directly that a Sylow subgroup of the groups in question contains at most five operators of order 2. Let s_1, s_2 represent any two operators of order 2 contained in the required group. There is clearly one and only one group in which the order of $s_1 s_2$ is either 8 or 9. The former of these was considered above. The order of $s_1 s_2$ cannot be 5 or 7 according to theorems I and VI. If this order is 6 each of the remaining operators of order 2 transforms the dihedral rotation group of order 12 into itself and H is the group of theorem II. Since the order of $s_1 s_2$ cannot be 2 for every value of s_1, s_2 it remains only to consider the cases when this order is either 3 or 4.

If $s_1 s_2$ is of order 4 the octic group (O_1) generated by s_1, s_2 involves five of the operators of order 2. Let s_3 represent one of the remaining four operators of this order. The transform (O_2) of O_1 with respect to s_3 is distinct from O_1 and has three operators of order 2 in common with O_1. We may suppose that s_1 is common to O_1 and O_2 while s_2 does not have this property. The transform (O_3) of O_2 with respect to s_2 contains s_3 and the remaining operator of order 2. These three octic groups are generated by their common operators and s_2, $s_2^{-1} s_3 s_2$, s_3 respectively. Hence they are transformed by G according to the symmetric group of order 6.

Since only two operators of order 2 in G correspond to an operator of order 2 in this symmetric group, the order of $s_2 s_3$ is either 3 or 6. The latter was considered above so that we may assume this order to be 3. The four-group which is common to O_1, O_2, O_3 together with $s_2 s_3$ generate the alternating group of order 12. As this is transformed into itself by s_2, the nine operators of order 2 generate the symmetric group of order 24. This symmetric group could therefore be defined as a group which contains exactly 9 operators of order 2 such that the order of the product of any two does not exceed 4 while there are at least two whose product is of this order.

If $s_1 s_2$ is of order 3 the group (D_1) generated by s_1, s_2 contains only three operators of order 2. Suppose that G contains another operator of order 2 s_3 which transforms D_1 into itself. As the dihedral rotation group $[D_1, s_3]$ contains operators of order 6 which are generated by two operators of order 2, the resulting group was considered above. Hence it may be assumed that no operator of order 2 contained in G transforms D_1 into itself. Let $s_3^{-1} D_1 s_3 = D_2$.

If D_1, D_2 have a common subgroup of order 3 they have no operator of order 2 in common and the 9 operators of order 2 are contained in three conjugate subgroups of order 6. As these are permuted by H according to the symmetric group of order 6, it follows that $s_1 s_3$ is an operator of order 3, which is commutative with the operators of order 3 in D_1. Hence H is the group of order 18 which is obtained by extending the non-cyclic group of order 9 by an operator transforming each of its operators into its inverse.

Suppose that G contains two operators of order 2 which constitute a complete set of conjugates. As these would generate an invariant four-group, G would involve an octic group and hence it would contain two operators of order 2 whose product would exceed 3. As these cases have been considered it follows that no two operators of order 2 form a complete set of conjugates under G in the cases which remain to be considered. Similarly, it is easy to see that three operators of order 2 could not form a complete set of conjugates in these cases since neither a four-group nor a symmetric group of order 6 can be invariant. If there were a complete set of four conjugates, they would be composed of two pairs of commutative operators and hence they would generate the octic group.

Five, six, or seven operators of order 2 could not form a complete set of conjugates without leading to groups already considered, since the remaining operators of order 2 could not all be invariant. If there were a complete set of eight conjugates of order 2, there would be an invariant operator of this order. Hence all the operators of order 2 would be contained in four conjugate four-groups. Each of these would transform one of the others into itself and hence G would contain the octic group. Hence the nine operators of order 2 in the cases which remain to be considered form one set of conjugates.

Hence H is isomorphic with a transitive substitution group (T) of degree 9 which has exactly 9 operators of order 2 and these also form a complete set of conjugates. The subgroup (K) which corresponds to the identity in T is of odd order and each of its operators is commutative with all the operators of H. The transitive groups of degree 9 are well known and it is easy to verify that only two of them are generated by such a set of 9 operators of order 2, viz., the two groups of order 18 which are of class 8. In verifying this statement it may be convenient to observe that T can contain no substitution of even order and of degree 9, and that the product of any two of its substitutions of order 2 is of order 3 and of degree 9.

The cases when H is simply isomorphic with T have been considered. We may therefore assume that K is not the identity. Suppose that s_3 is an operator of order 2 which is not in $[s_1, s_2]$. From T it follows that $s_3 s_1 s_2 s_3 = s_2 s_1 t$, where t is an operator of K. If $s_1 s_2$ and $s_1 s_3$ were commutative they would generate a group of order 9 and the 9 operators of order 2 would generate a group of order 18. Hence $s_1 s_2$, $s_1 s_3$ generate the non-commutative group of order 27 which involves 26 operators of order 3, since their commutator is an invariant operator of order 3. As s_1 is commutative with only three operators in this group of order 27, the nine operators of order 2 in question generate this group of order 54.

Hence, *there are exactly six groups which contain only* 9 *operators of order* 2 *and are generated by these operators.* The orders of these groups are: 16, 18, 18, 24, 24, 54. The first two are dihedral rotation groups, while the fourth is a symmetric group. The third is composed of the abelian non-cyclic group of order 9 and operators which transform each operator of this abelian group into its inverse. The fifth is the group of theorem II, § 1. The last is simply isomorphic with one of the transitive groups of degree 9. It contains 26 operators of order 3, 18 of order 6, and 9 of order 2 in addition to the identity.

§ 4. *Groups which Contain exactly eleven Operators of Order Two.*

Let s_1, s_2 represent two such operators. The order of $s_1 s_2$ cannot exceed 11. If it is either 10 or 11 the group is the dihedral rotation group of order 20 or 22. The order of $s_1 s_2$ could not be 9 according to theorem I § 1. If $s_1 s_2$ is of order 8, H is the group of theorem II § 1. This group is also completely defined by the facts that its order is 32 and that it contains exactly eleven operators of order 2, such that the order of the product of two of them is 8*. The order of $s_1 s_2$ cannot be 7 according to theorems I and VI.

If $s_1 s_2$ is of order 6, the dihedral rotation group generated by s_1, s_2 is invariant under H since a group of order 6 cannot involve more than three operators of order 2 and hence H cannot involve two such dihedral rotation groups which are conjugate under one of its operators of order 2. Let s_3 be any operator of order two contained in H but not in $[s_1, s_2]$. From the facts that $[s_1, s_2]$ contains a characteristic operator of order 2 and that H contains only 11 operators of this order it follows that $[s_1, s_2, s_3]$ is the group of theorem II § 1. Its order is 24 and it involves 9 operators of order 2. One of the remaining operators of order two (s_4) transforms this group of order 24 into itself and is commutative with 12 of its operators. As these can include only one operator of order 2 this isomorphism is again completely determined and $[s_1, s_2, s_3, s_4]$ is of order 48. This group is completely defined by its order and the fact that it involves just 11 operators of order 2 which generate it.

The case when s_1, s_2 is of order 5 need not be considered since it follows from theorem VI § 1 that H could not contain two dihedral rotation groups of order 10 which are conjugate under one of its operators of order 2. The dihedral rotation group of order 10 is therefore invariant and H is the direct product of this subgroup and a subgroup of order 2. In other words, H is the dihedral rotation group of order 20 since the direct product of the dihedral rotation group of order $2p$ and a group of order 2 is the dihedral rotation group of order $4p$.

It remains to consider the cases when the order of the product of any two operators of order two in G cannot exceed 4. If s_1s_2 is of order 4 the octic group $[s_1, s_2]$ may transform one of the remaining operators of order 2 into itself. In this case H is the only group of order 16 which contains exactly 11 operators of order 2. Since six operators of G are not in $[s_1, s_2]$ there is at least one pair of conjugates under this group. Let s_3 be one of such a pair. The subgroup of order 4 in $[s_1, s_2]$ which transforms s_3 into itself is cyclic, otherwise H would involve the group of order 8 and of type (1, 1, 1) as an invariant subgroup and hence would be the group of order 16 which has been considered.

---

* Quarterly Journal of Mathematics, Vol. 28, 1896, p. 241, No. 21.

If s_3 does not transform $[s_1, s_2]$ into itself it transforms it into an octic group which involves the same cyclic subgroup of order 4. An operator of order 2 (s_4) in this octic group but not in $[s_1, s_2]$ must transform $[s_1, s_2]$ into itself since G contains only 11 operators of order 2. Hence $[s_1, s_2]$ is transformed into itself either by s_3 or by s_4. In the latter case, G would contain an operator of order 2 not in $[s_1, s_2]$ and commutative with the operators of order 4 in $[s_1, s_2]$ since the order of the product of two operators of order 2 cannot exceed 4. We may therefore assume that s_3 transforms $[s_1, s_2]$ into itself. Hence $[s_1, s_2, s_3]$ is the group of order 16 which contains just 7 operators of order 2 and is generated by them.

Let s_5 represent any operator of order 2 in G but not in $[s_1, s_2, s_3]$. If s_5 did not transform $[s_1, s_2, s_3]$ into itself it would transform it into a group having an octic subgroup in common. An operator of order 2 in this group but not in the octic common subgroup would transform $[s_1, s_2, s_3]$ either into itself or into another group having this common octic subgroup. The latter case is impossible since the six operators of order 2 not in the common octic subgroup would constitute three pairs of conjugate operators which H would transform according to the symmetric group of order 6. As H would contain an invariant operator of order 2, viz., the characteristic operator of this order in the common octic subgroup, this would require that the product of two operators of order 2 in H would have an order larger than 4. Hence it may be assumed that s_5 transforms $[s_1, s_2, s_3]$ into itself.

If s_5 is so chosen as to be commutative with each operator of the quaternion group contained in $[s_1, s_2, s_3]$ while it tranforms each of the other operators into themselves multiplied by the operator of order 2 in this quaternion group the resulting group will involve 4 operators of order 2 and 12 of order 4 besides those of $[s_1, s_2, s_3]$. Hence this group of order 32 is distinct from the one considered above and is also generated by 11 operators of order 2. As there are only two groups of order 32 which contain exactly 11 operators of order 2 and are generated by these operators it has been proved that H is this group of order 32 whenever $s_1 s_2$ is of order 4 and no two of its operators of order 2 give a product of larger order.

It remains to consider the cases when the order of any two operators of order 2 in G does not exceed 3. If $s_1 s_2$ is of order 3 there must be at least two operators s_3, s_4 of order 2 which are commutative with $s_1 s_2$. If there are only two such operators they are conjugate under s_1 since G cannot contain the direct product of $[s_1, s_2]$ and a group of order 2. Hence $s_3 s_4$ is of order 3, commutative with $s_1 s_2$, and transformed into its inverse by s_1. This is impossible since s_1 would transform into their inverses all the operators of $[s_1 s_2, s_3 s_4]$. If $s_1 s_2$ were commutative with exactly five operators of order 2 one of these would also be commutative with s_1 and hence H would involve the direct product of $[s_1, s_2]$ and a subgroup of order 2.

As this is again impossible, it remains only to consider the case when $s_1 s_2$ is commutative with eight of the operators of order 2. Two of these s_3, s_4 would be conjugate under s_1 and hence $s_3 s_4$ would be transformed into its inverse by s_1. If $s_3 s_4$ is of order 3 this is the case considered above. If it is of order 2 it is commutative with s_1 and hence has also been considered. From this we conclude that *if a group contains exactly* 11 *operators of order* 2 *it must contain at least two such operators whose product is of an order which exceeds* 3, and that there are exactly six groups which involve only 11 operators of order 2 and are generated by them.

SUMMARY.

Combining the results of the article "*The Groups which Contain thirteen Operators of Order Two*" (Bulletin of the Am. Mathematical Society, Vol. XII, 1906, p. 289) with those of the present article, we arrive at the following table, where n represents the number of operators of order 2 in the groups which are generated by these n operators.

n	*Orders*					
1	2					
3	4	6				
5	8	10				
7	8	12	14	16		
9	16	18	18	24	24	54
11	16	20	22	32	32	48
13	24	26	40	48	72	

Concerning the Improper Definite Integral.[*]

By N. J. Lennes.

The following theorems suggested themselves in the course of writing out a detailed treatment of various topics in the elements of analysis. So far as known to the writer they are not, with one exception, to be found in the literature.

The aim of this note aside from pointing out that the improper definite integral possesses many of the important properties of the proper definite integral, is to show the impossibility of obtaining, in certain specified ways, stronger necessary conditions for the existence of the improper definite integral than have already been stated. It is, of course, no longer necessary to emphasize the importance of theorems to the effect that certain kinds of extensions of other theorems are impossible.

Definition: *A function is said to be bounded in the neighborhood of a point* $x = x_1$ *if there exists a pair of positive numbers* δ *and* M *such that for every* $x \mid |x - x_1| \leqq \delta$, $|f(x)| \leqq M$. *The function is unbounded in the neighborhood of a point if no such pair of numbers exist.*

If (a) a function $f(x)$ considered in the interval $a \dots b$ is bounded in the neighborhood of every point on this interval except $x = a_1$,

(b) $\int_x^b f(x)\,dx$ exists for every value of x on the segment $a \dots b$[†], (the segment not including its end-points),

(c) $\underset{x = a}{\angle} \int_x^b f(x)\,dx$ exists and is finite,

then the improper definite integral is said to exist in the interval $a \dots b$, or the integral is said to exist improperly at $x = a$.

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[*] Presented before the American Mathematical Society Dec., 1904.

[†] The definite integral is said to exist only for functions which are bounded on the interval under consideration. Hence (a) is redundant in the presence of (b).

Lemma: *For every function $f_1(x)$ which is unbounded in the neighborhood of $x = a$ there is a function $f_2(x)$ which is infinitesimal as x approaches a*[*] *such that $f_1(x) \,.\, f_2(x)$ is unbounded in the neighborhood of this point.* (It is to be understood that the functions $f_1(x)$ and $f_2(x)$ are considered only for values of x in an interval $a \dots b$, *i. e.* on one side of the point a.)

Proof: Decompose the interval $a \dots b$ into an infinite set of subintervals by means of an ordered set of points dense only at a.[†] Let this partition of $a \dots b$ be such that for some given positive number M there are values of x in the interval $x_1 \dots b$ for which $|f_1(x)| > 2\,M$ and in general there are values of x in $x_n \dots x_{n-1}$ for which $|f_1(x)| > 2^n M$. We define $f_2(x)$ as follows: $f_2(x) = 1$ for all values of x in the interval $x_1 \dots b_1$, $f_2(x) = \frac{1}{2}$ for all values of x in $x_2 \dots x_1$ excluding the point x_1, $f_2(x) = \frac{1}{n}$ in $x_n \dots x_{n-1}$ excluding the point x_{n-1}. Then there are values of x in $x_n \dots x_{n-1}$ for which $|f_1(x)| \,.\, f_2(x) > \frac{2^n}{n} M$ which shows $f_1(x) \,.\, f_2(x)$ unbounded in the neighborhood of $x = a$. Evidently $f_2(x)$ may be so chosen that $\frac{f_2(x)}{x-a}$ shall be monotonic increasing as x approaches a.[‡]

THEOREM I. *For every function $f_1(x)$ which is unbounded in the neighborhood of $x = a$ there exists a non-oscillating*[§] *function $f_2(x)$ such that $\underset{x=a}{\angle} \int_x^b f_2(x)dx$ exists and is finite while $(x - a)\, f_1(x) \,.\, f_2(x)$ is unbounded in the neighborhood of $x = a$.*

Proof: According to the lemma there exists a function $f_3(x)$ such that $\underset{x=a}{\angle}\, f_3(x) = 0$ while $f_3(x) \,.\, f_1(x)$ is unbounded. Consider a function $f_4(x) = \frac{f_3(x)}{x-a}$, f_4 being monotonic as x approaches a. Since $(x - a) f_4(x) \,.\, f_1(x) = f_3(x) \,.\, f_1(x)$, $(x - a) f_4(x) \,.\, f_1(x)$ is unbounded in the neighborhood of $x = a$. Let $x_1 \dots x_n \dots$

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[*] $f(x)$ is infinitesimal as x approaches a if $\underset{x=a}{\angle}\, f(x) = 0$.

[†] A set of points is said to be dense at a certain point a if there are points of the set, other than a, within every neighborhood of a.

[‡] In case $f_1(x) > 0$ for all values of x in same neighborhood of $x = a$ $f_2 = \frac{1}{\sqrt{f_1(x)}}$ or $f_2 = \frac{1}{\log f_1(x)}$ would satisfy the conditions imposed upon $f_2(x)$ by the lemma.

[§] A function $f(x)$ is said to be non-oscillating in an interval if as x increases in this interval the function either never decreases or never increases. This is different from a monotonic function which either constantly increases or constantly decreases.

be an ordered series of points in $a \ldots . b$ dense only at a such that $f_3(x) \, . \, f_1(x)$ is unbounded at this set. Then in the sequence

$$\begin{gathered}(x_1 - a) \, f_4(x_1), \qquad (x_2 - a) \, f_4(x_2), \; \ldots . \; (x_n - a) \, f_4(x_n) \; \ldots . \\ \underset{n \doteq \infty}{\angle} (x_n - a) \, f_4(x_n) = 0 \text{ since } \underset{x \doteq a}{\angle} (x - a) \, f_4(x) = 0.\end{gathered} \tag{1}$$

Hence there is a value of n, n_1 such that

$$|(x_1 - a) \, f_4(x_1)| \geqq 2 |(x_{n_1} - a) \, f_4(x_{n_1})|$$

and another value of n, n_2 such that

$$|(x_{n_1} - a) \, f_4(x_{\,1})| \geqq 2 |(x_{n_2} - a) \, f_4(x_{n_2}).$$

In general n_{m+1} is so chosen that

$$|(x_{n_m} - a) \, f_4(x_{n_m})| > 2 |(x_{n_{m+1}} - a) \, f_4(x_{n_{m+1}})| .$$

In this manner we select from the sequence (1) a set of terms forming the convergent series:

$$(x_1 - a) \, f_4(x_1) + (x_{n_1} - a) \, f_4(x_{n_1}) + \; \ldots . \; + (x_{n_m} - a) \, f_4(x_{n_m}) + \; \ldots . \tag{2}$$

We then obtain a function $f_2(x)$ as follows:

For the set of values of x, $x_{n_{m+1}} < x \leqq x_{n_m}$,

$$f_2(x) = | \, f_4(x_{n_m}) | .$$

Then

(*a*) $f_2(x)$ is non-oscillating since $| f_4(x_{n_m}) | < | f_4(x_{n_{m+1}}) |$,

(*b*) $(x - a) \; f_2(x) \, . \, f_1(x)$ is unbounded at the set $x_1, x_{n_1}, x_{n_2} \; \ldots . \; x_{n_m} \; \ldots .$ since for this set $f_2(x) \, . \, f_1 \equiv f_4(x) \, . \, f_1(x)$,

(*c*) $\underset{x = a}{\angle} \int_x^b f_2(x) dx = \sum_{m=0}^{\infty} (x_{n_{m+1}} - x_{n_m}) \, f_4(x_{n_m})$.

But the terms of this series are numerically smaller than the corresponding terms of the convergent series (2). Hence $\underset{x \doteq a}{\angle} \int_x^b f_2(x) dx$ exists and is finite.

In case $f_2(x)$ is non-oscillating and unbounded in the neighborhood of $x = a$ a well-known necessary condition that $\underset{x \doteq a}{\angle} \int_x^b f_2(x) dx$ shall exist and be finite is that $\underset{x \doteq a}{\angle} (x - a) \, f_2(x) = 0$. Theorem I may be regarded as showing this a strong necessary condition since according to it no function $f_1(x)$ of x can be pre-assigned which shall approach infinity slowly enough as x approaches a so that

$$\underset{x \doteq a}{\angle} (x-a) f_1(x) \,.\, f_2(x) = 0.^{*}$$

Lemma: *If* $\int_a^b f_1(x)dx$ *and* $\int_a^b f_2(x)dx$ *both exist, and if in the interval* $a \ldots . b$ $\left|\dfrac{f_1(x)}{f_2(x)}\right| \leqq M$ *and* $f_2(x)$ *does not change sign, then*

$$\left|\int_a^b f_1(x)dx\right| \leqq M \left|\int_a^b f_2(x)dx\right|.$$

Proof: From the definition of the definite integral and the obvious proposition that if $n_0, n_1 \ldots . n_n$ are numbers having the same sign and if

$$|v_0| \leqq |n_0|, \quad |v_1| \leqq |n_1|, \quad \ldots\ldots\ldots, \quad |v_n| \leqq |n_n|$$

then

$$\left|\sum_{i=1}^{n} v_i\right| \leqq \left|\sum_{i=1}^{n} n_i\right|$$

it follows that

$$\int_a^b |f_1(x)|\, dx \leqq M \int_a^b |f_2(x)|\, dx.$$

But

$$\left|\int_a^b f_1(x)dx\right| \leqq \int_a^b |f_1(x)|\, dx \quad \text{and} \quad \int_a^b |f_2(x)|\, dx = \left|\int_a^b f_2(x)dx\right|$$

therefore

$$\left|\int_a^b f_1(x)\, dx\right| \leqq M \left|\int_a^b f_2(x)\, dx\right|.$$

THEOREM II. *If (a)* $\dfrac{f_1(x)}{f_2(x)}$ *is bounded in some neighborhood of* $x = a$,

(b) $\int_x^b f_1(x)dx$ *and* $\int_x^b f_2(x)dx$ *both exist for every* x *in the segment* $a \ldots . b$,†

(c) $f_2(x)$ *does not change sign in some neighborhood of* $x = a$,

(d) $\underset{x \doteq a}{\angle} \int_x^b f_2(x)dx$ *is finite,*‡

then it follows that $\underset{x \doteq a}{\angle} \int_x^b f_1(x)dx$ *exists and is finite.*

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[*] This theorem was, in effect, stated and proved by A. Pringsheim in Mathematische Annalen, vol. 37 (1890), p. 591. The proof given here is however much shorter and simpler than that given by Pringsheim.

[†] The expression segment $a \ldots . b$ is used to include all values of x, $a < x < b$.

[‡] We notice that since under (c) of the hypothesis $f_2(x)$ does not change sign $\underset{x \doteq a}{\angle} \int_x^b f_2(x)dx$ cannot *fail* to exist, finite or infinite, for it follows from this that $\int_x^b f_2(x)dx$ is a non-oscillating function of x whence by a well-known theorem the limit exists.

Proof: Since by hypothesis $\underset{x=a}{\angle}\int_x^b f_2(x)dx$ exists and is finite it follows by an elementary theorem that for every ε there exist a δ_ε such that for every x_1 and x_2 on the segment $a\ldots b$ and also in a δ_ε neighborhood of $x = a$,

$$\left|\int_{x_1}^{x_2} f_2(x)dx\right| < \varepsilon .$$

Consider x_1 and x_2 in a neighborhood of $x = a$ in which $\left|\frac{f_1(x)}{f_2(x)}\right| \leqq M$ and $f_2(x)$ does not change sign.

Then by the lemma:

$$\left|\int_{x_1}^{x_2} f_1(x)dx\right| \leqq M \left|\int_{x_1}^{x_2} f_2(x)dx < M\varepsilon.\right.$$

Since $M\varepsilon$ can be made small at will by making ε small (M being independent of ε) it follows that $\underset{x\doteq a}{\angle}\int_x^b f_1(x)dx$ exists and is finite. This may be called the relative convergence theorem.

Definition: *If* $\frac{f_1(x)}{f_2(x)}$ *and* $\frac{f_2(x)}{f_1(x)}$ *are both bounded in the neighborhood of* $x = a_1$, $f_1(x)$ *and* $f_2(x)$ *being unbounded in that neighborhood, then we refer to these functions as having the same rank of infinity in the neighborhood of* $x = a$. *If* $\underset{x\doteq a}{\angle}\frac{f_1(x)}{f_2(x)} = K$, $K \neq 0$, *then they are said to be of the same order of infinity.*

THEOREM III. *If* (*a*) $f_1(x)$ *and* $f_2(x)$ *are of the same rank of infinity in the neighborhood of* $x = a$,

(*b*) $\int_x^b f_1(x)dx$ *and* $\int_x^b f_2(x)dx$ *both exist for every* x *on the segment* $a\ldots b$,

(*c*) *there is a neighborhood of* $x = a$ *in which* $f_2(x)$ *does not change sign,*

(*d*) $\underset{x=a}{\angle}\int_x^b f_2(x)$ *is infinite,* (*see note under Theorem II*)

then $\underset{x=a}{\angle}\int_x^b f_1(x)dx$ *cannot exist and be finite.*

Proof: This is a direct consequence of theorem II since by that theorem if $\underset{x=a}{\angle}\int_x^b f_1(x)dx$ were finite then $\underset{x=a}{\angle}\int_x^b f_2(x)dx$ would exist and be finite.

THEOREM IV. *If for a function $f_1(x)$ which does not change sign in the neighborhood of $x = a$ there exists a monotonic function $f_2(x)$ such that $f_1(x)$ and $f_2(x)$ have the same rank of infinity in the neighborhood of $x = a$, $\int_x^b f_1(x)dx$ and $\int_x^b f_2(x)dx$ both existing for any x on the segment $a \ldots . b$, then a necessary condition that $\underset{x \doteq a}{\angle} \int_x^b f_1(x)dx$ exists and is finite is that $\underset{x \doteq a}{\angle} (x - a) f_1(x) = 0$.*

Proof: By hypothesis $\underset{x \doteq a}{\angle} \int_x^b f_1(x)dx$ exists and is finite. Hence by theorem II $\underset{x \doteq a}{\angle} \int_x^b f_2(x)dx$ exists and is finite. Therefore by an elementary theorem (referred to on page 17) $\underset{x \doteq a}{\angle} (x - a) f_2(x) = 0$.

Since $\dfrac{f_1(x)}{f_2(x)}$ is bounded as x approaches a, *i. e.*, there exists an M such that $|f_1(x)| < M |f_2(x)|$, we have $|(x - a)| f_1(x)| < M |(x - a) f_2(x)|$.

But
$$\underset{x \doteq a}{\angle} M |(x - a) f_2(x)| = 0.$$

Therefore
$$\underset{x \doteq a}{\angle} |(x - a) f_1(x)| = 0 \text{ is } \underset{x \doteq 0}{\angle} (x - a) f_1(x) = 0.$$

THEOREM V. *For every function $f_1(x)$ defined in the interval $a \ldots . b$ there exists a function $f_2(x)$ such that*

(*a*) *$f_2(x)$ is continuous and does not change sign in a certain neighborhood of $x = a$,*

(*b*) *$\underset{x \doteq a}{\angle} \int_x^b f_2(x)dx$ exists and is finite,*

(*c*) *For x on a certain set $[x']$ $\underset{x' \doteq a}{\angle} \dfrac{f_1(x')}{f_2(x')} = 0$.*

Proof: (We describe the function $f_2(x)$ in the language of geometry).

Let $x_1', x_2', \ldots . x_n' \ldots .$ be a set of points in the interval $a \ldots . b$ dense only at a. Let $B_1, B_2 \ldots . B_n \ldots .$ be a set of numbers such that

$$B_n \cdot n \cdot f_1(x_n) \geqq 2\, B_{n+1} \cdot (n + 1) \cdot f_1(x_{n+1}) \quad (n = 1 \ldots . \infty).$$

On the x axis lay off segments $A_n C_n = B_n$* such that x_n are the middle points of the segments $A_n C_n$. The segments $A_n C_n$ must satisfy the further condition

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*There is no incongruity in speaking of segments equal to numbers if the statement is properly interpreted.

that they shall be small enough so as not to overlap. (This condition can be satisfied by making the numbers B_n small enough.) On the segments $A_n\ C_n$ as bases construct isosceles triangles on the positive side of the x-axis whose altitudes are $n\,.\,f_1(x_n)$. The measures of area of these triangles form a convergent series. Let $f_3(x)$ be a continuous function, monotonic and unbounded in the neighborhood of $x = a$, such that $\underset{x=a}{\angle}\int_x^b f_3(x)dx$ exists and is finite. $f_3(x)$ should also satisfy the condition that the curve which it represents has only one point in common with a side of one of the triangles described above. We then define $f_2(x)$ as the function represented by the following curve: 1st, those parts of the boundaries of the isosceles triangles just described which lie above the curve defined by $f_3(x)$. 2d, those parts of the curve defined by $f_3(x)$ which lie outside these triangles or on their boundary.

Obviously the function so defined has the properties specified in the theorem. From theorem V it follows that from the hypothesis only that $\underset{x=a}{\angle}\int_x^b f_1(x)\,dx$ exists and is finite, it is impossible to obtain any conclusion as to the boundedness of any function whatever $f_2(f_1(x))$ provided the values of f_2 depend only upon the values of $f_1(x)$ at a set of points whose limit points form a discrete closed set, and provided also f_2 is capable of becoming infinite for any values however large of the independent variable $f_1(x)$ at such points.

This is precisely what one would expect *a priori*, since the definite integral is a function of two dimensions while any condition in terms of boundedness of a function would necessarily be in terms of only one of these.

In case the definite integral exists *improperly* at a finite number of points and properly at all other points, the theorems just stated apply directly. In case the points at which the definite integral exists improperly are dense at a point a but discrete at all other points of a neighborhood including this point, we say the definite integral exists improperly at $x = a$ if $\underset{x=a}{\angle}\int_x^b f(x)dx$ exists and is finite, the limitand function being the improper definite integral in the interval $a \ldots . b$. Hereafter the sign $\int$ will include the improper as well as the proper definite integral, the term improper definite integral being used to denote an integral existing improperly at a set of points whose limit points form a discrete closed set.

The following theorems on the definite integral apply to the improper definite integral as well:

THEOREM VI. *If* $\int_a^b f(x)dx$ *exists then* $\int_a^c f(x)dx$ *and* $\int_c^b f(x)dx$ *both exist and* $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

The proof is obvious.

THEOREM VII. *If* $\int_a^b f(x)dx$ *exists then* $\int_a^x f(x)dx$ *is a continuous function of* x *for all values of* x *in the interval* $a \dots . b$.

Proof: The extension of the theorem to the improper definite integral follows immediately from the definition of the latter.

THEOREM VIII. *If* (*a*) $\int_a^b f_1(x) . f_2(x)dx$ *and* $\int_a^b f_1(x)$ *both exist,*

(*b*) $f_1(x)$ *does not change sign in* $a \dots . b$,

(*c*) $\overline{B}$ *and* $\underline{B}$ *are the least upper and the greatest lower bound respectively of* $f_2(x)$ *in* $a \dots . b$,

then
$$\overline{B} \int_a^b f_1(x) . dx \leqq \int_a^b f_1(x) . f_2(x)dx \leqq \overline{B} \int_a^b f_1(x) \quad \text{or}$$
$$\underline{B} \int_a^b f_1(x)dx \geqq \int_a^b f_1(x) . f_2(x)dx \geqq \overline{B} \int_a^b f_1(x)dx$$

Proof: The theorems $\overline{B} \int_a^b f_1(x)dx = \int_a^b \overline{B} f_1(x)dx$ and
$$\underline{B} \int_a^b f_1(x)dx = \int_a^b \underline{B} f_1(x)dx$$
$\overline{B}$ and $\underline{B}$ being constants, apply to the improper definite integral as well as to the proper since $\underset{x \doteq x_1}{\angle} af(x) = a \underset{x \doteq x_1}{\angle} f(x)$ when a is a constant.

Moreover in case $f_1(x)$ is always positive
$$\overline{B} f_1(x) \leqq f_1(x) . f_2(x) \leqq \overline{B} f_1(x).$$
Hence by the definitions of the definite integral and elementary considerations:
$$\int_a^b \underline{B} f_1(x)dx \leqq \int_a^b f_1(x) f_2(x)dx \leqq \int_a^b \overline{B} f_1(x)dx,$$
and therefore
$$\underline{B} \int_a^b f(x)dx \leqq \int_a^b f_1(x) f_2(x)dx \leqq \overline{B} \int_a^b f_1(x)dx.$$

If $f_1(x)$ is entirely negative it follows in the same manner that

$$\underline{B} \int_a^b f_2(x)dx \geqq \int_a^b f_1(x) \,.\, f_2(x)dx \geqq \overline{B} \int_a^b f_1(x) \,.$$

COROLLARY. (The first mean value theorem).

If in theorem VIII $f_2(x)$ is continuous there is a value of x, ξ in the interval $a \ldots . b$ such that

$$\int_a^b f_1(x) \,.\, f_2(x)dx = f(\xi) \,.\, \int_a^b f_1(x)dx.$$

THEOREM IX. (The second mean value theorem).

If the improper definite integral of $f_1(x)$ exists in $a \ldots . b$ and if $f_2(x)$ is non-oscillating in this interval, then there is a value of x, $x = \xi$ in the interval $a \ldots . b$ such that

$$\int_a^b f_1(x) \,.\, f_2(x)dx = f_2(a) \int_a^{\xi'} f_1(x)dx + f_2(b) \int_{\xi'}^b f_1(x)dx,$$

Proof: If $f_2(x)$ is non-oscillating it is integrable, consequently $\int_a^b f_1(x) \,.\, f_2(x)dx$ exists since the product of two integrable functions, one being integrable in the proper sense, can readily be shown to be integrable at least in the improper sense.

Suppose the integral of $f_1(x)$ to exist properly at every point of the interval $a \ldots . b$ except the point c. Consider a function $f_3(x)$ defined as follows in the interval $a \ldots . b$. $f_3(x) = f_2(x)$ in the interval $a \ldots . c - e$ (e being a small positive number) $f_3(x) = 0$ in the interval $c - e \ldots . c + e$ and $f_3(x) = f_1(x)$ in the interval $c + e \ldots . b$.

Assuming theorem IX where the integral exists in the proper sense we have that a ξ exists for every value of e however small such that

$$\int_a^b f_3(x) \,.\, f_1(x)dx = f_3(x) \int_a^{\xi} f_1(x)dx + f_3(b) \int_{\xi}^b f_1(x)dx.$$

As e approaches zero, ξ approaches one or more points as limit points. Let ξ' be such limit point. Then we assert

$$\int_a^b f_2(x) \,.\, f_1(x)dx = f_2(a) \int_a^{\xi'} f_1(x)dx + f_2(b) \int_{\xi'}^b f_1(x)dx. \quad (1)$$

Suppose this equation does not hold and that

$$\left| \int_a^b f_2(x) \,.\, f_1(x)dx - f_2(a) \int_a^{\xi'} f_1(x)dx - f_2(b) \int_{\xi'}^b f_1(x)dx \right| = \varepsilon. \quad (2)$$

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Since the definite integral is a continuous function of the limits of integration it follows that there exists a certain neighborhood of $x = \xi'$ such that for every value of x in this neighborhood

$$\left| f_2(a) \int_a^{\xi'} f_1(x)dx + f_2(b) \int^b f_1(x)dx - f_2(a) \int_a^x f(x)dx - f_2(b) \int_x^b f_1(x)dx \right| < \frac{\varepsilon}{2}.$$

Denote this neighborhood by δ_ϵ.

Likewise there is a certain neighborhood of $x = c$ which we denote by δ'_ϵ such that

$$\left| \int_{c + \delta_\epsilon}^{c + \delta_\epsilon} f_1(x) \cdot f_2(x)dx \right| < \frac{\varepsilon}{2}.$$

Therefore for all values of $\varepsilon < \delta_\epsilon$ the corresponding values of ξ cannot lie in the δ_ϵ neighborhood of $x = \xi'$ and consequently ξ' is not a value approached by ξ as e approaches zero, which is a contradiction with the hypothesis resulting from the assumption of equation (2). Hence equation (1) holds. In this manner we can prove the existence of a ξ' satisfying the conditions of the theorems if the definite integral of $f_1(x)$ exists properly everywhere in the interval except at a finite number of points. If the integral exists improperly at an infinite set of points whose limit points (first derived set) is a closed discrete set and if the improper definite integral as defined on page 19 exists in the interval $a \ldots b$ then the argument to show the existence of ξ' for such function is identical with the above.

The restriction on the set of points at which the integral exists improperly is rather artificial. Obviously the theorems can be extended to cover more general cases.

NOTE: For a more detailed discussion of the elementary properties of the improper definite integral see Infinitesimal Analysis (pp. 191-224), Veblen and Lennes. Published by John Wiley & Sons, New York. This book also contains a treatment of the improper definite integral in the case where it exists improperly at a more general set of points.

THE UNIVERSITY OF CHICAGO, *June* 15, 1905.

On the Congruence of Axes in a Bundle of Linear Line Complexes.

By O. P. Akers.

Introduction.

The congruence of axes contained in a three term group of linear line complexes has been quite exhaustively studied. Stahl* has treated in a synthetic manner, the (3, 2) congruence in considerable detail. Later† he discusses this congruence as a special case of a congruence of a higher order. We will take occasion later to refer to other papers on this subject. The discussion here will be with regard to some of the ruled surfaces contained in the congruence of axes. Some of the notations and methods of Study‡ will be used and some applications of these methods will be made to show the arrangement of the lines of the congruence on ruled surfaces contained in a linear complex.

If we choose the complex $\phi_1 = 0$ arbitrarily, $\phi_2 = 0$ in involution with it, and finally $\phi_3 = 0$ in involution with both, the coordinate complexes of the bundle may be written

$$p_{23} - e_1 p_{41} = 0, \qquad p_{31} - e_2 p_{42} = 0, \qquad p_{12} - e_3 p_{43} = 0,$$

and the bundle of complexes may be expressed by the equation

$$\sigma_1 (p_{23} - e_1 p_{41}) + \sigma_2 (p_{31} - e_2 p_{42}) + \sigma_3 (p_{12} - e_3 p_{43}) = 0. \qquad (1)$$

This equation defines ∞^2 complexes by varying the σ_1, σ_2, σ_3. Moreover, ∞^1 coaxial complexes, for given values of the σ_i, are defined by the equation

$$\sigma_1 (p_{23} - e_1 p_{41}) + \sigma_2 (p_{31} - e_2 p_{42}) + \sigma_3 (p_{12} - e_3 p_{43}) + p(\sigma_1 p_{41} + \sigma_2 p_{42} + \sigma_3 p_{43}) = 0, \qquad (2)$$

by giving p different values.

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* "Das Strahlensystem dritter Ordnung und zweiter Klasse"; Crelle, vol. 91, p. 1.

† "Das Strahlensystem vierter Ordnung und zweiter Klasse"; Crelle, vol. 97, p. 147.

‡ Geometrie der Dynamen; p. 461.

Since the axis is the same for every value of p, there is no restriction in taking the finite special complex to define the axis. If we solve the equation (2) for the coordinates of the axis, we obtain, when $e_1+e_2+e_3=0$ and $p(\sigma_1^2+\sigma_2^2+\sigma_3^2)=e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2$, the coordinates

$$p_{41}=\sigma_1, \qquad p_{42}=\sigma_2, \qquad p_{43}=\sigma_3,$$
$$p_{23}=(p-e_1)\sigma_1, \qquad p_{31}=(p-e_2)\sigma_2, \qquad p_{12}=(p-e_3)\sigma_3.$$

It will be assumed that $e_1>e_2>e_3$. This system of axes will be referred to as the (σ) system. The reciprocal (τ) system, defined by the system of three terms, every complex of which is in involution with every complex of the original bundle, has the coordinates

$$r_{41}=\tau_1, \qquad r_{42}=\tau_2, \qquad r_{43}=\tau_3,$$
$$r_{23}=-(p-e_1)\tau_1, \qquad r_{31}=-(p-e_2)\tau_2, \qquad r_{12}=-(p-e_3)\tau_3.$$

The (σ) system and the (τ) system will enter into the discussion in exactly the same way.

The σ_i may be regarded as point coordinates in the plane at infinity; and in this way the configuration can be mapped on the plane. To every point σ_i in the plane at infinity corresponds one finite line of the congruence and conversely; when a finite line is given, the values of σ_i are uniquely determined. The points of the plane at infinity and the lines of the congruence are in (1, 1) correspondence.

The congruence of axes in a bundle of linear line complexes contains fifteen singular planes and ten singular points. The dual case was first studied by Kummer ("Ueber die algebraischen Strahlensysteme, in's besondere ueber die der ersten und zweiten Ordnung." Berliner Abhandlungen, 1866). Five singular planes, one of which is the plane at infinity, are of the second order, the remaining ten are of the first order. Four singular points lie in each singular plane. The singular points on the absolute will be called cusps. The planes of the singular pencils which contain the tangents to the absolute at the cusps will be called cuspidal planes. The remaining six singular points will be called vertices, and the plane containing the pencil of lines passing through a vertex will be called a vertical plane.

The configuration in this congruence of axes is a particular case of the general (3, 2) congruence, the dual of the (2, 3) congruence discussed by Kummer. In the former case, four of the singular conics touch the fifth (the absolute), but in the latter case this is not, in general, true.

The coordinates of the cusps may be obtained from the relation $p(\sigma_1^2+\sigma_2^2+\sigma_3^2)=e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2$. When $\sigma_1^2+\sigma_2^2+\sigma_3^2=0$, and $e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2=0$, p is indeterminate. The coordinates of the common points of the two conics are

$$\sigma_1:\sigma_2:\sigma_3=\sqrt{e_2-e_3}:\sqrt{e_3-e_1}:\sqrt{e_1-e_2},$$

the signs of the radicals being permuted for the different points. Through these four points pass ∞^1 axes lying in a plane. If we write the equation of the plane determined by two lines of the congruence which pass through a cusp, we obtain the following,

$$s:t:u:v=\sqrt{e_2-e_3}\sqrt{e_3-e_1}\sqrt{e_1-e_2}:\sqrt{e_2-e_3}:\sqrt{e_3-e_1}:\sqrt{e_1-e_2},$$

which are the coordinates of the cuspidal planes in the (σ) congruence. The corresponding cuspidal planes in the (τ) congruence have the coordinates

$$s:t:u:v=-\sqrt{e_2-e_3}\sqrt{e_3-e_1}\sqrt{e_1-e_2}:\sqrt{e_2-e_3}:\sqrt{e_3-e_1}:\sqrt{e_1-e_2}.$$

If the coordinates of the cusp C_1 are $\sqrt{e_2-e_3}:\sqrt{e_3-e_1}:\sqrt{e_1-e_2}$ and those of C_2 are $\sqrt{e_2-e_3}:-\sqrt{e_3-e_1}:\sqrt{e_1-e_2}$, then the equation of the line joining C_1 and C_2 is $\sqrt{e_1-e_2}\,\sigma_1-\sqrt{e_2-e_3}\,\sigma_3=0$. To every point of the line C_1C_2 corresponds one finite line of the congruence. All these form a plane pencil in some plane passing through C_1C_2. To find the plane and the vertex of the pencil, find the lines corresponding to any two points on C_1C_2. These two lines have a plane and a point in common. The equations of this plane and point are found to be

$$\sqrt{e_1-e_2}\,x+\sqrt{e_2-e_3}\,z=0,\qquad s+\sqrt{e_1-e_2}\sqrt{e_2-e_3}\,u=0,$$

in the (σ) congruence or

$$\sqrt{e_1-e_2}\,x+\sqrt{e_2-e_3}\,z=0,\qquad s-\sqrt{e_1-e_2}\sqrt{e_2-e_3}\,u=0,$$

in the (τ) congruence. There are two real vertices and vertical planes. The six vertical planes intersect by twos, on the principal rays, lines passing through the vertices of the common self conjugate triangle with reference to the absolute and the curve $e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2=0$. These three lines are mutually perpendicular and pass through a common point. It is convenient to take this point for origin (0, 0, 0, 1). Each principal ray contains two vertices. Each cuspidal plane contains one cusp and three vertices, and each vertical plane contains two cusps and two vertices. The vertices and planes are the same for both congruences, but differently paired. The arrangement of the singular points and the singular planes is readily seen from the tables. C_i are the cusps; c_i are the corresponding cuspidal planes. S_i, S_i' are the vertices lying on the principal ray i; s_i, s_i' are the vertical planes intersecting on i. Each singular plane contains the four singular points of the same column.

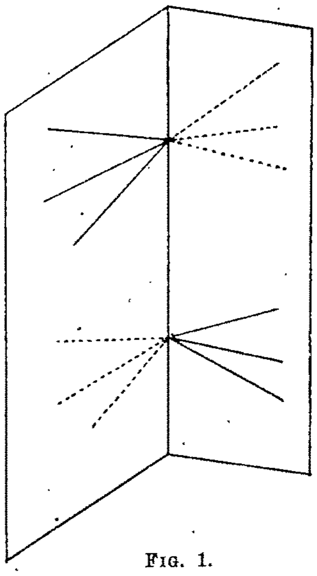

FIG. 1.

c_1	c_2	c_3	c_4
C_1	C_2	C_3	C_4
S_1	S_1'	S_1	S_1'
S_2	S_2	S_2'	S_2'
S_3'	S_3	S_3'	S_3

s_1	s_1'	s_2	s_2'	s_3	s_3'
C_1	C_3	C_1	C_2	C_1	C_2
C_2	C_4	C_3	C_4	C_4	C_3
S_1	S_1	S_2	S_2	S_3	S_3
S_1'	S_1'	S_2'	S_2'	S_3'	S_3'

CHAPTER I.

§1. *Parameter Surfaces.*

The equations of any line may be written

$$\left.\begin{aligned} p_{43}y + p_{24}z + p_{32}w &= 0 \\ p_{34}x + p_{41}z + p_{13}w &= 0. \end{aligned}\right\} \qquad (1)$$

When p is defined by $p(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2$, any line of the congruence may be expressed by

$$\left.\begin{aligned} \sigma_3 y - \sigma_2 z - (p - e_1)\,\sigma_1 w &= 0 \\ -\sigma_3 x + \sigma_1 z - (p - e_2)\,\sigma_2 w &= 0. \end{aligned}\right\} \qquad (2)$$

If the last equations be solved for $\sigma_1 : \sigma_2 : \sigma_3$, the result is

$$\frac{\sigma_1}{xz + a_2 yw} = \frac{\sigma_2}{yz - a_1 xw} = \frac{\sigma_3}{z^2 + a_1 a_2 w^2}, \qquad (3)$$

where $a_i = p - e_i$. Substituting the value of p in equation (3), we have the equation of the parameter surface

$$a_1x^2 + a_2y^2 + a_3z^2 + a_1a_2a_3w^2 = 0, \tag{4}$$

or in plane coordinates

$$a_2a_3s^2 + a_3a_1t^2 + a_1a_2u^2 + v^2 = 0. \tag{5}$$

One system of lines on this surface belongs to the (σ) congruence and the other system belongs to the (τ) congruence for every value of p.* To each value of p corresponds a parameter surface. The equation

$$p(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2$$

represents a pencil of conics in the plane $w = 0$, having the cusps for basis points. One point will fix a conic; the same point fixes a finite line of the congruence, hence every line of the congruence lies on just one parameter surface. For $p = \infty$, $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0$, hence the absolute is included as a degraded case of a parameter surface. The plane $w = 0$, is therefore a singular plane of the second order; through any point in it pass two tangents to the absolute and one finite line. Between the latter and the point exist a (1, 1) correspondence. For points on the absolute, the line is also tangent to the absolute, and at the cusps the correspondence breaks down, as every line of the cuspidal pencil corresponds to the same point, namely, the cusp. If the coordinates of an arbitrary point (x', y', z', w') be substituted in equation (4), we obtain a cubic equation in p, which shows that three parameter surfaces of the system can be drawn through each point of space. The tangent planes at (x', y', z', w') each contain two generators, one generator belongs to the (σ) system and the other belongs to the (τ) system. Since a line of (σ) uniquely fixes the parameter surface to which it belongs, through each point of space pass three lines of the congruence and the congruence is of the order three. The equation of the parameter surface (4) may be factored into two expressions and written

$$\left.\begin{aligned} \lambda(\sqrt{a_2}\,y + i\sqrt{a_3}\,z) &= \sqrt{a_1a_2a_3}\,w + i\sqrt{a_1}\,x \\ \lambda(\sqrt{a_1a_2a_3}\,w - i\sqrt{a_1}\,x) &= -\sqrt{a_2}\,y + i\sqrt{a_3}\,z \end{aligned}\right\} \qquad \text{A}$$

or

$$\left.\begin{aligned} \mu(\sqrt{a_2}\,y - i\sqrt{a_3}\,z) &= \sqrt{a_1a_2a_3}\,w + i\sqrt{a_1}\,x \\ \mu(\sqrt{a_1a_2a_3}\,w - i\sqrt{a_1}\,x) &= -\sqrt{a_2}\,y - i\sqrt{a_3}\,z. \end{aligned}\right\} \qquad \text{B}$$

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* See Waelsch, "Ueber eine Strahlencongruenz beim Hyperboloid." Wiener Berichte, vol. 95, pp. 781–801, 1887; also Stahl, Crelle, vol. 91.

If (x_1, y_1, z_1, w_1) are the coordinates of a fixed point lying on the hyperboloid, we may eliminate λ and μ from these two groups of equations and get the relations

$$\begin{aligned}\sqrt{a_1 a_2 a_3}\,(\sqrt{a_2}\,y_1 + i\sqrt{a_3}\,z_1)\,w + i\sqrt{a_1}\,(\sqrt{a_2}\,y_1 + i\sqrt{a_3}\,z_1)\,x \\ - (a_2\sqrt{a_1 a_3}\,w_1 + i\sqrt{a_2 a_1}\,x_1)\,y - (ia_3\sqrt{a_1 a_2}\,w_1 - \sqrt{a_1 a_3}\,x_1)\,z = 0,\end{aligned}$$

and

$$\begin{aligned}\sqrt{a_2}\,(\sqrt{a_1 a_2 a_3}\,w_1 - i\sqrt{a_1}\,x_1)\,y - i\sqrt{a_3}\,(\sqrt{a_1 a_2 a_3}\,w_1 - i\sqrt{a_1}\,x_1)\,z \\ - \sqrt{a_1 a_2 a_3}\,(\sqrt{a_2}\,y_1 - i\sqrt{a_3}\,z_1)\,w + i\sqrt{a_1}\,(\sqrt{a_2}\,y_1 - i\sqrt{a_3}\,z_1)\,x = 0.\end{aligned}$$

If we express the line coordinates from these last two equations, we have

$$\begin{aligned}\sigma_1 : \sigma_2 : \sigma_3 = x^2 + a_2 a_3 w^2 : xy + a_3 wz : xz - a_2 wy = xy - a_3 wz : \\ y^2 + a_3 a_1 w^2 : yz + a_1 wx = xz + a_2 wy : yz - a_1 wx : z^2 + a_1 a_2 w^2, \qquad (6)\end{aligned}$$

which shows the relation among the σ_i for an arbitrary point (x, y, z, w). If this point lies in the plane $w = 0$, but not on the absolute, then two roots of equation (4) become infinite. If the point is on the absolute but not at a cusp, all three roots are infinite. If however, the point is at a cusp, p is indeterminate and (6) furnishes a pencil of lines through this point. When the coordinates of the real vertices, $1 : 0 : \mp\sqrt{e_1 - e_2}\sqrt{e_2 - e_3} : 0$, are substituted we get the lines of the vertical pencils. Similarly, equation (5) gives the relation

$$\begin{aligned}\sigma_1 : \sigma_2 : \sigma_3 = - a_3 u^2 - a_2 v^2 : a_3 tu + sv : a_2 tv - su = a_3 tu - sv : \\ - a_1 v^2 - a_3 t^2 : a_1 uv + st = a_2 tv + su : a_1 uv - st : - a_2 t^2 - a_1 u^2. \qquad (7)\end{aligned}$$

If the plane (s, t, u, v) touches the absolute, but not at a cusp, then equation (5) has one infinite root; but if the point of contact is a cusp and the plane is not a cuspidal plane, both roots are infinite. If the plane is a cuspidal plane of (τ), (7) furnishes ∞^1 lines of (σ) which envelope a conic, which touches the absolute, the tangent at the cusp being the common tangent.

§2. *Focal Surface.*

The locus of the ∞^2 points of intersection of consecutive lines of the congruence is called the focal surface. Evidently the equation of this locus may be obtained by considering the condition that the equation of the parameter surface

(4), as a cubic in p, shall have two equal roots.* Hence writing the discriminant of that equation we have

$$4\{(e_2 e_3 + e_3 e_1 + e_1 e_2) w^2 + x^2 + y^2 + z^2\}^3 + 27 w^2 (e_1 e_2 e_3 w^2 + e_1 x^2 + e_2 y^2 + e_3 z^2)^2 = 0. \qquad (8)$$

Similarly equation (5) gives

$$4(t^2 + u^2 + v^2)(s^2 + e_2 e_3 t^2 + e_3 e_1 u^2 + e_1 e_2 v^2) - (e_1 t^2 + e_2 u^2 + e_3 v^2)^2 = 0, \qquad (9)$$

which shows that the focal surface is of order six and class four, which agrees with the results obtained by Kummer's formulas.

The focal surface is the same for both congruences. It is the complete envelope of the parameter surfaces, and it is also the locus of the limiting points and limiting planes. Each point of the surface is a focal point for one line, *i. e.*, two consecutive lines of the congruence and a limiting point for another line of the congruence. Each plane of the focal surface is a focal plane for one line, *i. e.*, two consecutive lines of the congruence, and a limiting plane for another line of the congruence, namely, for the third line of the congruence which passes through the focus. The intersection of the ($p = 0$) hyperboloid with its director sphere is a cuspidal curve on the focal surface.† Moreover, the plane $w = 0$ intersects the focal surface in the absolute, which is also a cuspidal curve. These curves are the loci of points for which the tangent planes to the focal surface and the tangent planes to the parameter surface $p = 0$, coincide; *i. e.*, from points on these curves passes only one line of the congruence counted three times. Since in the plane $w = 0$, the tangents to the absolute are lines of the congruence, the focal points and limiting points coincide on the absolute and we have lines of six point contact.

From the equation of the focal surface, in plane coordinates, it is seen that a cuspidal plane is tangent to the surface in a conic C_1, which is the locus of the point of contact of this plane with the parameter surface. Moreover, the same cuspidal plane intersects the focal surface in another conic C_2. The two conics have a common tangent, namely, the tangent to the absolute at a cusp. We have seen that the cuspidal plane contains a pencil of minimal lines which belongs to (σ), and whose vertex is the cusp; and that it contains, also, ∞^1 finite, but imaginary lines of (τ) which envelope the conic C_2. The focal points of the minimal lines lie on the conic C_1. Their limiting points are on C_2. At the cusp, one focal point and a limiting point coincide. It is interesting to notice that the (τ) lines, in this plane, touch the focal surface in three points. However these are not all three focal points, but at one of the points of tangency the limiting points have come into coincidence. Plainly, when the (τ) line passes through the cusp, the three points of tangency coincide and we have a line o six-point contact, mentioned above. The (σ) congruence contains ∞^2 bitangents to the focal surface, Fig. 3. The points F_1 and F_2 are the focal points, L_1 and L

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* Hyde, "On a surface of the sixth order which is touched by the axes of all screws reciprocal to three given screws." *Ann. Math.*, series 2, vol. 2, p. 179.

Gruenwald, "Zur Veranschaulichung des Schraubenbuendels" (*Schloemilch's Zeitschrift*, vol. 49, p. 211).

Joly, "Geometry of a three system of screws." *Trans. Royal Acad.*, vol. 32, Sec. A, pp. 239–270.

† Stahl, *Crelle*, vol. 97, p. 146, sec. 8.

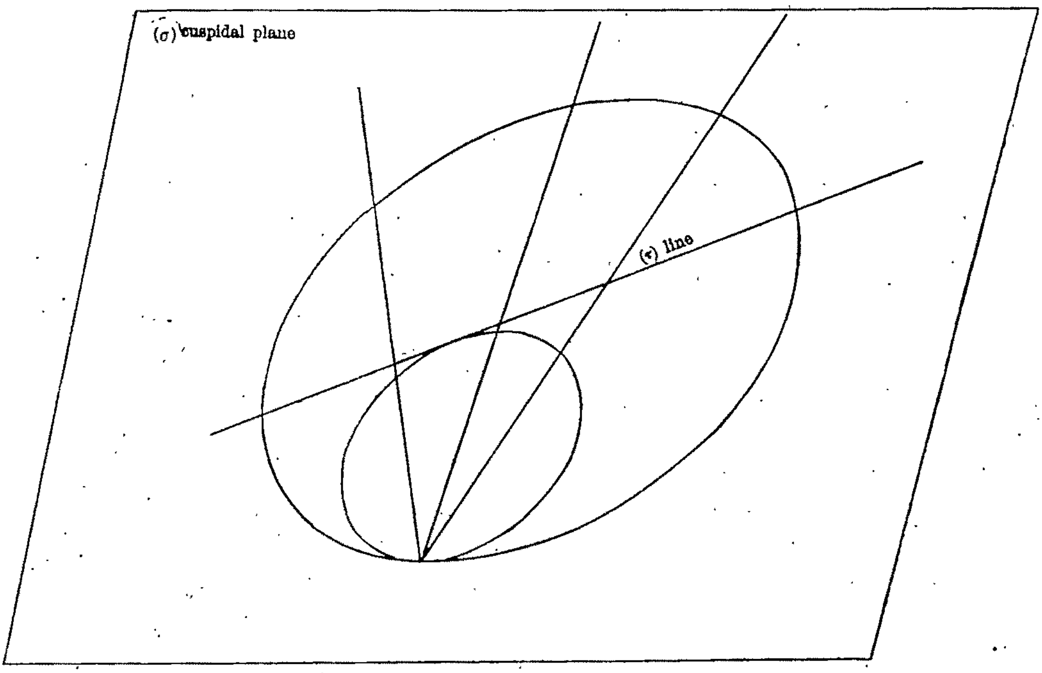

FIG. 2.

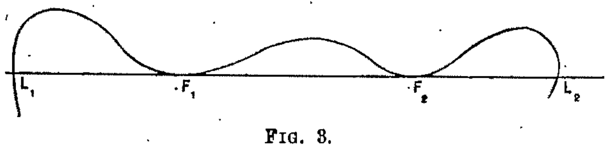

FIG. 3.

are the limiting points. F_1 and F_2 may coincide as in figure 4, and we have a line of four point contact with the focal surface. The locus of the ∞^1 lines o four-point contact is a surface of order eight.* The ∞^1 lines of triple contact

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*Study, *l. c.*, p. 485.

with the focal surface, which belong to the (σ) congruence, are the (σ) lines in the (τ) cuspidal planes, Fig. 5.

The vertical planes touch the focal surface in one circle and cut it in another circle. The infinite points on the one circle are the cusps lying in the plane. The tangents at the cusps are the lines of (σ) cut from the cuspidal planes, hence the two circles are concentric. Any plane section of the focal surface has the following Pluecker numbers:

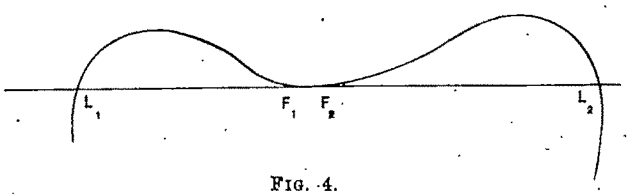

FIG. 4.

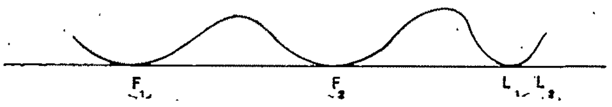

FIG. 5.

$$m=6,\quad n=12,\quad \delta=0,\quad k=6,\quad i=24,\quad \tau=27,\quad p=4.$$

For a tangent plane the numbers are

$$m=6,\quad n=10,\quad \delta=1,\quad k=6,\quad i=18,\quad \tau=15,\quad p=3.$$

From an arbitrary point, the tangent cone to the surface has the numbers

$$m=12,\quad n=4,\quad \delta=28,\quad k=24,\quad \tau=0,\quad i=0,\quad p=3,$$

and for a point in a singular (double) plane

$$m=10,\quad n=4,\quad \delta=16,\quad k=18,\quad \tau=1,\quad i=0,\quad p=2.$$

The remaining special cases can be easily determined.

CHAPTER II.

§3, S_5. *Surfaces in a Special Linear Complex.*

Since a (1, 1) correspondence exists between the points σ_1, σ_2, σ_3, in the plane $w=0$, and the finite lines of the congruence passing through them, any ruled surface contained in the congruence is completely defined by the relation among the σ_i, the equation of the section of the surface by the plane $w = 0$. In particular, if such a surface is contained in a linear complex

$$\Sigma a_{ik} p_{lm} = 0,$$

and we substitute the coordinates of the lines of (σ) for the p_{lm}, we obtain a cubic curve in terms of the σ_i, of the form

$$\frac{e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \frac{a\sigma_1 + b\sigma_2 + c\sigma_3}{a'\sigma + b'\sigma_2 + c'\sigma_3}. \tag{10}$$

Hence the image of any ruled surface contained in a linear complex is a c_3 passing through each of the four cusps. Conversely, since every such cubic curve contains five independent constants, it follows that the ∞^5 linear complexes and the ∞^5 cubic curves through the cusps are in (1, 1) correspondence. When $\Sigma a_{ik} a_{lm} = 0$, the complex becomes special; the ruled surface now consists of the lines of the congruence which cut the line a_{ik}. The c_3 cuts the absolute in two points apart from the cusps. The necessary and sufficient condition that the tangents to the absolute at these points should intersect on the curve is $\Sigma a_{ik} a_{lm} = 0$. Hence the complex is special when and only when the tangents to the absolute at the non-singular points in which it intersects the c_3, intersect on the latter.

Given any curve of order n in the plane $w = 0$. To determine the order of the ruled surface having this curve for section in the plane $w = 0$, it is sufficient to count the number of its generators which intersect an arbitrary line p_{ik}. The latter is the axis of a special linear complex, and the image of the lines of the congruence belonging to it is a c_3. The points of c_n on c_3, not at the cusps, correspond to the lines common to both ruled surfaces, and thus determine the number of lines of the ruled surface which cut the given line. If c_n goes through the cusp O_i, r_i times, the order of the ruled surface is $3n - r_1 - r_2 - r_3 - r_4$.*

We shall first study the ruled surfaces in a special linear complex. Through each point of an arbitrary line pass three lines of the congruence, and each plane through the arbitrary line contains two lines of the congruence, therefore, a ruled surface of order five is generated. The order of the double curve on such a surface is $6 - p$, where p is the genus of the surface. The triple line pierces the plane $w = 0$ in a point P, from which the tangents to the absolute are generators on the S_5. The remaining curve of section in this plane is a cubic,

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*Study, *l. c.*, p. 463.

which passes through P, the four cusps and the points of contact of the two tangents to the absolute from P. This surface will, in general, not have a double generator, hence the double curve is of order five. The residual curve to the triple line is a conic cutting the triple line once. This type of surface is $B\ II$ in Schwarz's classification,[*] Snyder's (23).[†] The surface can be generated by the lines joining corresponding points of a (3, 2) correspondence between a straight line and a conic having a double self-corresponding element at their point of intersection.

Through the point P pass ∞^3 finite lines, and in general, to each one of these lines corresponds an S_5. The lines of the congruence can be arranged on ∞^1 such S_5. We will consider a pencil of ∞^1 directrices in any finite plane. In order to get the ruled surfaces having such a pencil for directrices, first consider the directrix through $(a, b, 1)$ in $z = 0$ and through (λ, μ, ν) in $w = 0$. Its coordinates are

$$\begin{array}{lll} p_{12} = a\mu - b\lambda, & p_{13} = a\nu, & p_{14} = -\lambda, \\ p_{23} = b\nu, & p_{42} = \mu, & p_{34} = -\nu. \end{array}$$

A general line of the congruence will cut this line when

$$\begin{aligned} &(b\lambda - a\mu)\,\sigma_3 - \nu\,(p - e_3)\,\sigma_3 + a\nu\sigma_2 - \mu\,(p - e_2)\,\sigma_2 - \lambda \\ &\qquad\qquad (p - e_1)\,\sigma_1 - b\nu\sigma_1 = 0. \qquad (11) \end{aligned}$$

An equation of the form

$$pa + qb + r = 0 \qquad (p,\ q,\ r \text{ constants})$$

represents a line in the plane $z = 0$. If by means of this equation b be eliminated from the general cubic, a linear pencil of cubics

$$c_3 + ac_3' = 0$$

results, which has nine basis points; P, the four cusps C_i, the two points of contact of the tangents from P to the absolute, T_i, and two others, through each of which passes a line of the congruence, common to all the ∞^1 ruled surfaces whose directrices are lines of the plane pencil. They are the lines of the congruence in the plane of the pencil. Hence to find the basis points of the pencil of cubics, find the two points in which the line of intersection of the plane $w = 0$ and the plane of the pencil of the directrices cuts any cubic of the pencil.

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[*] "Ueber die geradlinigen Flaechen fuenften Grades." *Crelle*, vol. 67, pp. 23–57, (1867).

[†] Bulletin, vol. 8, pp. 293–296.

The point of intersection of the two lines of (σ) in the plane of the pencil of directrices is a point on the double conic, therefore, the double conics of all the ∞^1 ruled surfaces go through a common point. All the S_5 have five generators in common, two in the plane of the pencil and three through the vertex of the pencil.

If the plane of the pencil P, $pa+qb+r=0$, does not go through a cusp nor a vertex, each S_5 has just one line belonging to each cuspidal pencil and to each vertical pencil, namely, the line joining the cusp or vertex to the point in which the directrix pierces its plane. Similarly, if the plane is not tangent to any of the conics in the (τ) cuspidal planes, no lines in these planes will belong to more than one S_5 of the pencil. The only lines which belong to more than one S_5 are the finite lines of S_5 through P, the two tangents to the absolute through P, and the two lines of (σ) in the plane of the pencil of directrices. Given P and the line of the pencil in the plane $w=0$, any S_5 is uniquely determined when one line of it is known. From the formula, the S_5 corresponding to the infinite line through P breaks up into a cubic surface and the plane $w=0$, counted twice. Except for particular positions of the plane, the other S_5 will not be factorable.

§4. *Other Congruences having same Focal Surface.*

Through each point of an arbitrary line l, in the plane $w=0$, passes one finite line of the (σ) congruence. These lines of the congruence form a cylindroid whose double directrix is a finite line passing through the pole of l with respect to the absolute. If the line l passes through a cusp, the surface of the third order breaks down; the factors being a quadric and the cuspidal plane at the cusp. If l passes through two cusps, the surface consists of the two corresponding cuspidal planes and the vertical plane passing through l. To return to the case in which the line l passes through one cusp, say, $\sqrt{e_2-e_3}:\sqrt{e_3-e_1}:\sqrt{e_1-e_2}$; we may write the equation of this line in the form

$$\sqrt{e_1-e_2}\,(\sigma_1-\lambda\sigma_2)=\sigma_3\left(\sqrt{e_2-e_3}-\lambda\sqrt{e_3-e_1}\right). \tag{12}$$

The lines of (σ) which cut l form a paraboloid. One line of the paraboloid must lie in the plane at infinity. It must be the tangent at k, the point in which l cuts the absolute again. All the lines of the second generation must cut this tangent at k. By letting λ take all the values, a new $(3, 2)$ congruence (π_1) is determined. l is itself a line of the new congruence, therefore the plane at infinity is a singular plane of the first order for the new congruence. In the same way we get three other congruences (π_2), (π_3), (π_4) for each of the three remaining cusps. The cusps are the vertices of the four pencils. Let P be any point in the plane at infinity. Through P can be drawn two tangents to the absolute t_1, t_2. Let t_i touch the absolute at T_i. Let C_1 be one of the cusps. $T_1\ C_1$ determines a paraboloid. One system of lines on this paraboloid belongs to the (σ) congruence. t_1 belongs to (σ), but every generator that cuts t_1 belongs to (π_1). In particular, through P passes a generator g_1 of the paraboloid λ_1. Similarly, through P passes a generator g_2 of the paraboloid λ_2, determined by the line $T_2\ C_1$. The third line of (π_1) is the line PC_1. The plane determined by g_1, g_2, cannot contain a third line of the congruence, since it would be a tangent plane to a third (σ) paraboloid and therefore contain a generator of the other system, but the other system belongs to (σ), hence the plane would contain three lines of (σ) which is impossible. If g_1 be chosen for a directrix of a ruled surface in (σ), the paraboloid λ_1 must be a factor of the S_5. The residual surface must be a cubic S_3 having g_1 as a double directrix, since through each point of g_1 pass three lines of (σ), one of which belongs to λ_1. The trace of the paraboloid λ_1 in the plane at infinity is t_1 and $T_1\ C_1$. Two generators of the cubic surface pass through P, namely, t_2 and the finite line of (σ) through P. Since the section of every ruled surface of (σ), in the plane at infinity, which belongs to a special linear complex must pass through all four cusps, it follows that the section made on S_3 by this plane consists of t_2 and a conic passing through P, the three remaining cusps and T_1. Similarly, if g_2 be chosen for the directrix, another S_3 will result. The two conics will each pass through P, C_2, C_3, C_4. The new S_3 will contain t_1 instead of t_2. This configuration exists for every point of the plane at infinity. Moreover a similar system of paraboloids and of S_3 exists for the same point P with regard to the congruences (π_2), (π_3), (π_4). The association of the S_3 with the conics is given by the formula $3n - r_1 - r_2 - r_3 - r_4$.

Since g_i is tangent to the focal surface in two points, it cuts the same surface in two other points. These points are pinch points on the S_3. The S_3 are not cylindroids.

From the manner in which the paraboloids are generated, it is seen that the lines of the congruence can be arranged on either system of ∞^1 such surfaces. Hence, the common focal surface of the two reciprocal or complementary congruences (σ) and (τ) is also the focal surface of the four other congruences (π_1), (π_2), (π_3), (π_4), which are of order three and class two.

The (σ) and (τ) cuspidal planes at C_i are singular planes of the second order in (π_i).

A plane section of the focal surface is a curve of order six, and has six cusps, but no double points. From Pluecker's numbers, we saw, it has twenty-seven bitangents. In this plane are two lines of each of the six congruences; they are all bitangents. The other fifteen are lines of intersection with the fifteen double planes. Similarly, the twenty-eight double edges of a cone from any point are the 3×6 lines of the six congruences passing through the point and the six lines to the vertices, and four to the cusps.

§5. S_5 *With Triple Directrix and Double Generator.*

The directrix l of an S_5 will cut the focal surface in six points. Since each of the six points is a focal point for a line of the congruence, it is a pinch point on the S_5. Through l can be drawn four planes which are tangent to the focal surface. Since each of these four planes is a focal plane for a line of the congruence, the focal point in this plane is also a pinch point on the S_5, making in all, ten pinch points on the S_5, which agrees with Lueroth's formula. The four pinch points in the tangent planes through l are on the double conic. Two of the pinch points on l coincide when l becomes tangent to the focal surface, but when l is such a tangent, two of the tangent planes to the focal surface through l coincide. Therefore, two of the generators of the S_5 which pass through the coincident pinch points of l have also a common point on the double conic, or we have a double generator on the S_5. Hence, among the ∞^2 lines through an arbitrary point P, are ∞^1 tangent lines to the focal surface, to each of which corresponds a ruled surface of order five having a double generator. In general, the cone of tangents to the focal surface from any point P is of order twelve, it has twenty-four cuspidal edges and twenty-eight double edges. When P is in a singular (double) plane, we saw that the cone was of order ten. For certain positions the cone may be of lower order. On those ruled surfaces S_5, which have double generators the corresponding cubics in the plane $w = 0$ have nodes. The c_3 corresponding to a bundle of lines form a net; their basis points are, when the point P is taken in the plane $w = 0$, P, C_1, C_2, C_3, C_4, T_1, T_2. The locus of the node is the locus of the trace of the double generator in the plane $w = 0$. We may obtain the equation of this locus by forming the Jacobian of the net of curves represented by the equation (11). Equation (11) may be written in the form

$$a(\nu\sigma_2 - \mu\sigma_3)\Sigma + b(\lambda\sigma_3 - \nu\sigma_1)\Sigma + (\nu e_3\sigma_3 + \mu e_2\sigma_2 + \lambda e_1\sigma_1)\Sigma \\ - (\nu\sigma_3 + \mu\sigma_2 + \lambda\sigma_1)(e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2) = 0, \qquad (13)$$

where $\Sigma = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$. Its Jacobian is

$$\begin{vmatrix} 2\sigma_1(\nu\sigma_2 - \mu\sigma_3) & \nu\Sigma + 2\sigma_2(\nu\sigma_2 - \mu\sigma_3) & -\mu\Sigma + 2\sigma_3(\nu\sigma_2 - \mu\sigma_3) \\ -\nu\Sigma + 2\sigma_1(\lambda\sigma_3 - \nu\sigma_1) & 2\sigma_2(\lambda\sigma_3 - \nu\sigma_1) & \lambda\Sigma + 2\sigma_3(\lambda\sigma_3 - \nu\sigma_1) \\ (\qquad\qquad) & (\qquad\qquad) & (\qquad\qquad) \end{vmatrix}$$

This locus consists of the absolute and a quartic curve having a double point at P, passing through the cusps, and touching the absolute at T_1, T_2.

We may verify these results with some special cases. Let P be taken at $\sigma_1 = 0$, $\sigma_2 = 0$. Through this point pass ∞^2 lines, all parallel to the principal ray $x = 0$, $y = 0$. Any one of these lines $x + aw = 0$, $y + bw = 0$, is the axis of a special complex. The condition that this line cuts the line

$$\begin{aligned} \sigma_3 y - \sigma_2 z - (p - e_1)\sigma_1 w &= 0 \\ -\sigma_3 x + \sigma_1 z - (p - e_2)\sigma_2 w &= 0 \end{aligned}$$

is $a\sigma_2 + (p - e_3)\sigma_3 - b\sigma_1 = 0$, or solving for p

$$\frac{b\sigma_1 - a\sigma_2 + e_3\sigma_3}{\sigma_3} = \frac{e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \tag{14}$$

This cubic is the trace in the plane $w = 0$, of the S_5 having $x + aw = 0$, $y + bw = 0$, for the triple line. It passes through $(0, 0, 1)$ and cuts the absolute at $(i, \pm 1, 0)$. The tangents to the absolute at these points pass through $(0, 0, 1)$. In general this cubic is of genus one. On putting $\sigma_1 = m\sigma_2$ in the equation and letting $\sigma_3 = 1$, we have

$$(mb - a)(m^2 + 1)\sigma_2^2 + [(e_3 - e_1)m^2 + (e_3 - e_2)]\sigma_2 + mb - a = 0. \tag{15}$$

The discriminant of this equation is

$$\begin{aligned} [4b^2 - (e_3 - e_1)^2]m^4 - 8abm^3 + [4b^2 + 4a^2 - 2(e_3 - e_1)(e_3 - e_2)]m \\ - 8abm + 4a^2 - (e_3 - e_2)^2 = 0. \end{aligned} \tag{16}$$

For a given value of m, the two roots of (15) define the points in which the line $\sigma_1 = m\sigma_2$ cuts the cubic. When these two points coincide, the line is a tangent, hence the directions of the four tangents to c_3 from P are the roots of equation (16). If equation (16) has a double root, c_3 has a node, *i. e.* c_3 has a node when the m-discriminant of (16) vanishes. This is an expression of order twelve in a and b, which agrees with the result obtained above. The cubic now becomes

$$a\sigma_2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - b\sigma_1(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \sigma_3[(e_1 - e_3)\sigma_1^2 + (e_2 - e_3)\sigma_2^2] = 0. \tag{17}$$

The Jacobian is

$$\begin{vmatrix} 2\sigma_1\sigma_2 & -\Sigma - 2\sigma_1^2 & 2\sigma_1\sigma_3(e_1 - e_3) \\ \Sigma + 2\sigma_2^2 & -2\sigma_1\sigma_2 & 2\sigma_2\sigma_3(e_2 - e_3) \\ 2\sigma_2\sigma_3 & -2\sigma_1\sigma_3 & (e_1 - e_3)\sigma_1^2 + (e_2 - e_3)\sigma_2^2 \end{vmatrix}$$

or

$$\Sigma\,[\sigma_1^2(e_1 - e_3) + \sigma_2^2(e_2 - e_3)]\,[(\sigma_1^2 + \sigma_2^2 - \sigma_3^2)] = 0.$$

Thus the Jacobian breaks up into the absolute, the two lines through P and the cusps, and another conic touching the absolute at the points of contact of the tangents; namely $\sigma_1^2 + \sigma_2^2 - \sigma_3^2 = 0$. The lines of (σ), which cut a line joining two cusps, are the lines of a pencil whose vertex is a vertex. The lines of (σ) which cut the absolute, are its tangents, in the plane $w = 0$; hence, the locus of double rays, which are finite lines not belonging to a vertical pencil, must have the conic $\sigma_1^2 + \sigma_2^2 - \sigma_3^2 = 0$ for its trace in the plane $w = 0$. The ruled surface formed by double generators is of order six. It is rational since a simple conic lies on it. The two tangents from P to the absolute are each double generators. The residual double curve is of order eight.

If we put the coefficients of the two highest powers of m in equation (16) equal to zero, the equation has a double root at infinity. The corresponding value of σ_2 in (15) is $\sigma_2 = 0$. From (14) the corresponding values of σ_1 are $\sigma_1(\sigma_1 \pm 1)^2 = 0$. But $\sigma_1 = 0$ is at the origin, hence the coordinates of the double points are $\sigma_1 = \pm 1$, $\sigma_2 = 0$, $\sigma_3 = 1$. The equations of the double generator of the S_5 having the line $x = 0$, $2y = \pm(e_1 - e_3)w$, for triple line, are therefore

$$y \pm (p - e_1)\,w = 0 \quad x \pm z = 0; \text{ but}$$
$$p = (e_3 - e_1)\,/\,2, \text{ therefore } 2y = \pm(e_1 - e_3)\,w.$$

Between

$$\frac{(e_1 - e_3)\sigma_1 + 2e_3\sigma_3}{2\sigma_3} = \frac{e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \frac{\sigma_3 y - \sigma_2 z + e_1\sigma_1 w}{\sigma_1 w}$$
$$= \frac{-\sigma_3 x + \sigma_1 z + e_2\sigma_2 w}{\sigma_2 w} = \frac{\sigma_2 x - \sigma_1 x + e_3\sigma_3 w}{\sigma_3 w}$$

it is possible to eliminate the σ_i rationally. The result will be an S_5 having a triple directrix, a distinct double generator and a double conic.

If we put $x = 0$, $2y + (e_1 - e_3)\,w = 0$, $w = 1$, in the equation of the focal surface, we have

$$(-3e_2^2 + 4z^2)^3 + 27\,(e_2^3 + 4e_3 z^2)^2 = 0.$$

This equation has z^2 for a factor, therefore the directrix touches the focal surface. The double generator intersects the directrix at $(o, (e_3 - e_2), o, 2)$, *i. e.* at the point in which the directrix touches the focal surface. The plane containing the directrix and the nodal generator is given by the equation $2y + (e_1 - e_3) w = 0$, but this is the tangent plane to the focal surface at the point $(o, (e_3 - e_2), o, 2)$, as is easily verified.

Among the ∞^2 S_5, associated with the lines passing through an arbitrary point P, the ∞^1 S_5, whose directrices are the tangents from P to the focal surface, have three lines in common. The lines of the congruence can all be arranged on such ruled surfaces, each having a double generator.

§6. *Directrix l a Line of* (σ).

When the directrix is a line of (σ), through each of its points pass two lines of (σ) besides l itself. There can be no double curve on the locus apart from l, since from each point of such a curve, would pass two generators, each of which would cut l. That would be impossible because only two lines of (σ) lie in a plane. Every generator of an S_5 cuts three other generators, hence l must count for two generators as well as a double directrix. The surface has, therefore a four-fold line in l. It is unicursal. This type of ruled surface is a particular form of Schwarz's A I, but not mentioned, Snyder's (31). The curve on any plane section of the ruled surface containing two generators is a nodal cubic having its mode at the point of intersection of the generators (*i. e.* where l pierces the plane). Such a surface can be generated by a nodal cubic and a straight line through the node in (1, 2) correspondence without self corresponding elements. The points at which one of the variable generators coincides with l are the foci; the points at which two generators, distinct from l, coincide are limiting points. These are the pinch points on the ruled surface. In the former case l touches the focal surface, in the latter, it intersects the focal surface. The foci are the points on l which correspond to the two values of the parameter of the cubic at the node. The limiting points are the branch points of the (1, 2) correspondence. The limiting points divide the line l, considered as a directrix of a surface, into real and imaginary segments. If the foci are in the real segment, the corresponding cubic in the plane $w = 0$ is crunodal. If the foci are in the imaginary segment the cubic is acnodal. If l touches the focal surface in a point of the curve of order sixteen, mentioned by Study (*l. c.* p. 485), the two foci coincide, and the cubic curve in the plane $w = 0$, has a cusp.

The tangents to the absolute at the variable points of intersection of the c_3 with the absolute, are the generators of the S_5 in the plane $w = 0$. These tangents intersect at the node of the c_3.

If the point P describes any locus in the plane $w = 0$, the corresponding S_5 will comprise the entire (σ) congruence, but each line will belong to several surfaces. If the locus is of order n going r times through C_i, each line of (σ) will belong to $3n - r_1 - r_2 - r_3 - r_4$ different ruled surfaces.

§7. *Degraded* S_5.

Some interesting cases of degraded S_5 may be obtained by letting the directrix l take special positions. When l lies in a singular plane or passes through a singular point, the corresponding S_5 is reducible. In particular, the discriminant curve, $I^3 - 27J^2 = 0$, of equation (16), in the plane $z = 0$, will contain the traces of the vertical planes through (x, y) in the plane, as factors, each counted twice. Any line from a point of either of these lines to $P \equiv (0, 0, 1, 0)$, will be the directrix of a degraded S_5, since the vertical plane will appear as a factor.

The vertical planes cut the plane $w = 0$ in the lines joining two cusps. These lines are therefore factors of the respective associated cubic curves. The other factor is a conic passing through the remaining two cusps, and through the points of tangency of the tangents from P to the absolute. Let Q be a point on the trace of a vertical plane in $z = 0$. The line PQ determines a ruled surface and therefore determines a conic in the plane $w = 0$. The conics and points Q are therefore projective and the corresponding pair of double points (intersection of the line and the conic) are in involution. The ruled surface of the fourth order determined by this conic, has PQ for a double line. It has also a double conic cutting it; it is always unicursal. This surface can be generated by the points of a conic and a line cutting it, in (2, 2) correspondence, the point of intersection being a double self-corresponding element. It is Cayley's VII.

For certain points on the discriminant curve, the corresponding cubic curve will have a cusp; these points are at the intersection of $I = 0$, and $J = 0$. They are also cusps on the discriminant curve. For a finite number of points the discriminant will be a perfect square. This means that the cubic curve will have two double points, hence become reducible.

The lines of the congruence can be arranged on the ∞^1 S_4, whose directrices lie in a vertical plane and pass through an arbitrary point in this plane.

If the directrix be an arbitrary line through a vertex, the vertical plane occurs as a factor, and we have remaining an S_4. This surface has a triple directrix, and no simple directrix. This is Cayley's IX. In this case the lines of the congruence can be arranged again on ∞^1 S_4 whose directrices are a plane pencil of lines through the vertex. Every line of the congruence belongs to just one such surface, except the focal axis passing through the vertex. It belongs to every ruled surface of the system. Two lines of the vertical pencil whose vertex lies on l, belong to S_4, but a different pair for each line l of the pencil. No line of the pencil can belong to two ruled surfaces, because they are determined from the points of intersection of the trace of the plane in the plane $w=0$ and the conic, the trace of the S_4 in the plane $w=0$. All such conics, intersect in C_1, C_2, T_1, T_2, hence they can have no other points in common.

When the directrix l lies in the plane $w=0$, a conic C_2 through l, the pole of l with respect to the absolute, and the four cusps, is uniquely determined. l and C_2 make a degraded c_3 corresponding to a degraded ruled surface of order five. l is the simple directrix of a cylindroid, as has been pointed out, whose double line is the finite line of (τ) passing through L. Thus to any line l, in the plane $w=0$, is a definite C_2, but conversely if C_2 be given, there are ∞^1 lines l associated with it, any one of which, will, with C_2, determine a degraded S_5. The simple directrices of these cylindroids envelop a conic in the plane $w=0$, which is the polar conic of C_2 with respect to the absolute. C_2 is also the trace of the ∞^1 double directrices of the family of cylindroids. These lines form the (τ) system of the parameter surface through L. In this way every line of the (σ) congruence is counted twice. By interchanging the (σ) and the (τ), the other system of the parameter surface would be the double directrices; the simple directrices would be the same as before.

Now let l become tangent to the absolute; the polar of l becomes the point of contact, *i. e.* the simple directrix and the double directrix coincide and we have an S_3 known as Cayley's S_3. A plane through l contains one generator of each of the S_3 ((σ) and (τ)) corresponding to l. The locus of the point of intersection of the generators is a plane cubic curve having a node at the point of tangency of l and the absolute. The S_3 of the (σ) congruence intersects the S_3 of the (τ) congruence in a plane cubic curve.

Besides, there are a finite number of other ruled surfaces. If l passes through two vertices, the two vertical planes associated with these vertices are factors of the S_5. If l is a principal ray, it is a line of (τ) as well as (σ), and the lines of (σ) which cut it from a cylindroid whose directrix in the plane $w = 0$ is the side of the self-polar triangle opposite the vertex which is pierced by the pincipal ray. If l is a focal axis, the two cuspidal planes, which intersect on this focal axis, are factors of the S_5.

§8. *Other Loci.*

Besides the surfaces which comprise the arrangement of the lines of the congruence, some others present themselves, which are worthy of mention.

From §5, we see that associated with any point P in the plane at infinity is a ruled surface formed of the double generators. The trace of this surface, in the plane at infinity, is the Jacobian of the cubic and the two tangents from P to the absolute. The Jacobian is of order four and passes once through each cusp. Therefore the surface is of order eight. It is of genus two. The total nodal curve is of order nineteen, or seventeen besides the double generators (tangents to the absolute from P). The finite line of (σ) through P is a generator.

In every pencil of planes, having a finite axis, is just one plane such that the lines of (σ) lying in this plane intersect on the axis. This is the point in which the axis is cut by the double conic belonging to the S_5 determined by the given axis as directrix. Corresponding to the bundle of lines through a given point we would have ∞^2 such points of intersection forming a surface. To find the form of this locus, take any directrix line l through P and let it cut the double conic in Q. Pass a plane π through l and a (σ) line σ_p, through P. π contains another line g_1 of (σ). These two lines intersect at R on the double conic. Hence, the double conics for each directrix l in the plane π must pass through R. Moreover, through R passes a third line g_2 of (σ). The pencil P in the plane (σ_p, g_2) will define ∞^1 S_5 whose double conics pass through R. Then to each point of σ_p corresponds two sets, each containing an infinity of conics. Hence the three lines of (σ) through P are double lines on the surface. The point P is a triple point. Each line through P pierces the surface in one other point apart from P, hence the locus of the point is a surface of order four, having three double lines through R, hence, a Steiner surface of class three. When P lies in a singular (double) plane, for example, the plane at infinity, this plane must be a factor of the surface, since two (σ) lines through any point in this plane lie in the plane. In this case the surface is of order three, having the finite line of (σ) through P for a double line, hence it is a ruled surface. The tangents to the absolute from P, are simple lines on the S_3, and the simple directrix lies in the plane at infinity. It is the polar of P with regard to the absolute, hence the surface is a cylindroid.

Every point of every double conic associated with P lies on the Steiner surface associated with P. When a plane cuts a conic from a Steiner surface, it must be a tangent plane, hence the planes of the double conics envelop the same Steiner surface.

Through each point of the double conic, belonging to the S_5, pass three lines of the congruence. Two of these lines belong to the S_5. The third line generates a new surface of order five, on which the conic is simple. It belongs to another linear complex, usually not special, and contains a double generator.

CHAPTER III.

Surfaces in a Non-Special Linear Complex.

In the former chapter, the ruled surfaces which composed the arrangement of the lines of the congruence, belonged to a special linear complex. In this chapter we shall investigate the surfaces formed by the lines of the congruence, when the complex to which they belong is non-special. The cubic curve in the plane at infinity will not pass through the point of intersection of the tangents to the absolute at the variable points of intersection of this c_3 with the absolute (Chapter II, §3). The new surfaces cannot have a rectilinear directrix.

§9. c_3 *in Plane at Infinity.*

The line coordinates of a line joining the point $(-a, -b, 1)$ in the plane $z=0$ and the point (λ, μ, ν), in the plane $w=0$, are

$$\begin{array}{lll} p_{12}=-a\mu+b\lambda, & p_{13}=-a\nu, & p_{14}=-\lambda, \\ p_{23}=-b\nu, & p_{42}=\mu, & p_{34}=-\nu. \end{array}$$

This line is the axis of a special linear complex whose equation is

$$A\equiv\nu p_{12}+(a\mu-b\lambda)p_{34}-\mu p_{13}+a\nu p_{43}+b\nu p_{14}+\lambda p_{23}=0.$$

The pencil of coaxial complexes, having the same line for axis is

$$A+k(\nu p_{34}-\mu p_{43}+\lambda p_{14})=0.$$

The corresponding pencil of cubic curves in the plane $w = 0$ is obtained by substituting for p_{ik}, the coordinates of the axis:

$$\nu(p - e_3)\sigma_3 - (a\mu - b\lambda + k\nu)\sigma_3 + \mu(p - e_2)\sigma_2 + (a\nu - k\mu)\sigma_2$$
$$- (b\nu + k\lambda)\sigma_1 + \lambda(p - e_1)\sigma_1 = 0.$$

and eliminating p. We have then

$$(\lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3)\frac{e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \nu e_3\sigma_3 + (a\mu - b\lambda + k\nu)\sigma_3 + \mu e_2\sigma_2$$
$$- (a\nu - k\mu)\sigma_2 + (b\nu + k\lambda)\sigma_1 + \lambda e_1\sigma_1 = 0. \tag{18}$$

This is a linear system of $\infty^3\, c_3$, obtained by varying a, b, k. All the curves of the system pass through C_4, the four cusps, and T_1, T_2, the points of contact of the tangents to the absolute from $P \equiv (\lambda, \mu, \nu)$ in the plane $w = 0$. If we regard a, b as constant, we obtain ∞^1 coaxial complexes:

$$(\lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3)(e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2) - (\lambda e_1\sigma_1 + \mu e_2\sigma_2 + \nu e_3\sigma_3)(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$
$$- (a\mu - b\lambda)\sigma_3(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + (a\nu\sigma_2 - b\nu\sigma_1)(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$
$$= k(\lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3)(\sigma_1^2 + \sigma_2^2 + \sigma_3^2). \tag{19}$$

When k is equal to zero, we have the same cubic curve that we studied in connection with the special complex. The basis points of the system of cubic curves are the four cusps, the intersection of $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0$, and $e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2 = 0$; the points of T_1, T_2, the intersection of $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0$ and $\lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3 = 0$, each counted twice, since all the c_3 touch each other at T_1, T_2; the point determined by the intersection of $(\lambda e_1 + b\nu)\sigma_1 + (\mu e_2 - a\nu)\sigma_2 + (\nu e_3 + a\mu - b\lambda)\sigma_3 = 0$ and $\lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3 = 0$. When $k = \infty$, the corresponding c_3 factors into the absolute and the polar line of P with regard to the absolute; this factorable c_3 has two nodes. The pencil $c_3 + kc_3' = 0$ has ten other nodal cubics.

§10. S_5 *in Congruence.*

To each cubic curve in the plane at infinity corresponds an S_5. The cubic will not, in general, have a node. Hence the genus of the ruled surface is one, and the nodal curve is a space curve of order five. This type is Schwarz's BI, Snyder's (22). Every generator cuts the nodal curve three times. From an arbitrary point upon it, the twisted quintic will project into a quartic cone, and the two generators from the point of projection will be double edges upon it, hence the quintic curve is of genus one. An arbitrary line cuts an S_5 in five points. The axis of the complex cuts two generators in the plane $w = 0$. The lines of a non-special linear complex which cut the axis, must cut it at right angles, hence, the finite lines of the S_5 which cut the axis must pierce the plane $w = 0$ on the line $\lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3 = 0$, but there is only one such point on the cubic curve apart from T_1, T_2. This is evidently true for every arbitrary line in space, hence, only one finite line of the congruence cuts an arbitrary line (not belonging to (τ)) at right angles.

The tangent lines to the absolute from P, each count twice, as lines of the congruence intersecting the axis of the complex, hence the axis has four point contact with S_5 at $(\lambda, \mu, \nu, 0)$. The lines which cut the axis are independent of k; they are common to all the complexes of the coaxial pencil. Each of the nodal curves of the surfaces corresponding to the ∞^1 values of k, will pass through P. The lines of the congruence can be arranged on these surfaces, such that every line will belong to one complex.

The nodal quintic curves through P will generate a surface, which is a factor of the k-envelope of the family of S_5.

The two generators of the S_5 in the plane at infinity, cut the cubic in three points. One of these points is T_4, the point of tangency to the absolute. The nodal curves of the S_5 pass through T_4. The two generators in the plane $w = 0$ lie on the surface formed by the nodal quintics of the various S_5. Similarly the finite line cutting the axis lies on this surface.

Every ruled surface contained in a linear complex has an algebraic asymptotic line whose order is equal to the class of an arbitrary plane section of the surface.* In our case the plane section is a quintic having five double points, its class is therefore ten. The number of pinch points on the surface is ten. They are included among the intersections of the nodal curve and the focal surface.

§11. S_5 *Having Double Generator.*

When the cubic curve in the plane at infinity has a node, the S_5 must have a double generator, and consequently only six pinch points. All the pinch points are on the focal surface, hence, when the ruled surface has a double generator, the nodal curve must touch the focal surface twice, namely, at the foci of the double generator.

The genus of the ruled surface is now zero. The nodal curve contains the double generator as a factor; since the double curve has a triple point (Schwarz, *l. c.*, p. 12), the twisted quintic has a node, therefore, it is of genus zero.

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* Picard, "Mémoire sur une application de la théorie des complexes linéaires à l' étude des surfaces et des courbes gauches." Annales de l'Ecole Normale, 1877. Snyder, Bulletin Amer. Math. Society, vol. 5, p. 343 ff.

A plane section of the S_5 has six double points, from which the order of the complex asymptotic line is eight.

The cubic curve associated with P, and with parameter k may be written

$$\begin{aligned} a(\mu\sigma_3 - \nu\sigma_2)(\sigma_1^2+\sigma_2^2+\sigma_3^2) + b(\nu\sigma_1 - \lambda\sigma_3)(\sigma_1^2+\sigma_2^2+\sigma_3^2) \\ - (\lambda\sigma_1+\mu\sigma_2+\nu\sigma_3)(e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2) \\ + (\lambda e_1\sigma_1+\mu e_2\sigma_2+\nu e_3\sigma_3)(\sigma_1^2+\sigma_2^2+\sigma_3^2) \\ + k(\lambda\sigma_1+\mu\sigma_2+\nu\sigma_3)(\sigma_1^2+\sigma_2^2+\sigma_3^2) = 0. \end{aligned}$$

The coefficients of a and b are the same as they were for the system of cubics in the special linear complex, hence the Jacobian of the system contains the absolute as a factor. It is linear in k and passes through the basis points of the system of cubics. Any finite line of the congruence is a double generator on a ruled surface of order five belonging to a complex whose axis passes through P, and it belongs to just one such complex, since the Jacobian, which is the trace of the double generators in $w=0$ is linear in k; and k is therefore uniquely determined by the double generator. The coefficient of k in the equation of the pencil of Jacobians is $3(\lambda^2+\mu^2+\nu^2)(\sigma_1^2+\sigma_2^2+\sigma_3^2)^2$. Therefore the curves touch each other at the points T_1, T_2. They do not pass through P. The cusps and T_i are all basis points.

§12. *Degraded S_5.*

By a proper choice of the axis of the complex, the ruled surfaces may become reducible. If a plane, which contains a pencil of lines of the congruence, belongs to the complex, then the S_5 is composed of this plane and a ruled surface of order four. The nodal curve on such a surface is a twisted cubic. The type of this surface is Salmon's VI, Cayley's X. An example of the above mentioned surfaces may be obtained by taking for the axis of the coaxial system of complexes, the line through a vertex and perpendicular to the vertical plane. The vertical plane will now belong to every complex of the coaxial system, and we have ∞^1 S_4 which comprise the lines of the congruence. These surfaces cannot have a double generator. If the vertical plane belongs to the complex, and the vertex is not on the axis, we have a single surface of the fourth order as described above.

In a similar way, if two plane pencils of the congruence belong to the complex, the S_5 will be composed of these two planes and a ruled surface of order three. In order that two such planes belong to the coaxial system of complexes, it is necessary that the vertices of the pencils of lines lie on the axis of the coaxial system and that the planes of the two pencils be perpendicular to this axis. This will not, in general, be true for our configuration. However, a single surface of this kind may be formed when a principal ray is a line of the complex and the vertical planes belong to the same complex.

If a line of (τ) be chosen as axis of the complex, the cylindroid, whose directrices are the axis of the complex and the polar of P with regard to the absolute, is common to all the complexes of the coaxial system. Therefore the system (∞^1) hyperboloids comprise the lines of the congruence. The line coordinates for the point (λ, μ, ν) in the plane $w = 0$ are

$$\begin{aligned} p_{12} &= -(p' - e_3)\nu, & p_{13} &= (p' - e_2)\mu, & p_{14} &= -\lambda, \\ p_{23} &= -(p' - e_1)\lambda, & p_{42} &= \mu, & p_{34} &= -\nu. \end{aligned}$$

The corresponding cubic in the plane $w = 0$ is, when

$$p = \frac{e_1\lambda^2 + e_2\mu^2 + e_3\nu^2}{\lambda^2 + \mu^2 + \nu^2}$$

$$(\lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3)\,(p - p' + k) = 0.$$

The factors of this equation are the polar of P with regard to the absolute and a conic passing through the four cusps. When $k = 0$, we have the relation $p = p'$, which is the condition that a line of (σ) cut a line of (τ).

CORNELL UNIVERSITY, *May* 1, 1905.

On Septic Scrolls Having a Rectilinear Directrix.

By Charles H. Sisam.

Introduction.

Methods of Classifying Ruled Surfaces.

1. The most obvious method of classifying ruled surfaces is by means of their (x, y, z, w) equations. This method was successfully used by Salmon[*] in his classification of cubics and quartics. For surfaces of higher degree, however, this method is too cumbersome to be employed.

2. Salmon[†] also considered ruled surfaces geometrically as the loci of lines which cut three fixed curves on them. Many ruled surfaces are not, however, the complete locus of lines cutting three curves on them; and this method gives very little information concerning the separate components when the locus is composite.

3. An important variation of the above is obtained by considering the locus of a line which cuts one curve twice and another once. Such surfaces will frequently be referred to in this article as "scrolls of bisecants," the second curve, however, always being straight line. The degree of such a scroll of bisecants is,[‡] in general, $h - m'\frac{(m'-1)}{2} + 1/2\,(m-m')\,(m-m'-1)$ where m is the order of the curve, m' the number of points in which the curve meets the straight line, and h the number of its apparent double points. The straight line is an $h - \frac{m'(m'-1)}{2}$-fold line and the curve an $(m-m'-1)$-fold curve on the surface.[§]

4. A further variation I shall occasionally refer to is the locus of a line cutting a curve thrice. The degree of such a surface is[||] $(m-2)\;[h - 1/6m\,(m-1)]$.

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[*] Geometry of three dimensions, 4th Ed., pp. 485–8, and 512–22.

[†] On a class of ruled surfaces, Cambridge and Dublin Math. Journal, Vol. VIII, p. 45.

[‡] Salmon, Geometry of three dimensions, 4th Ed., p. 481.

[§] The condition that the surface be composite will be obtained later.

[||] Salmon, l. b., l. d., p. 432.

5. Schwarz,[*] in his excellent classification of quintic scrolls, considers them as the loci of the lines of intersection of corresponding planes of two developables between which a one to one correspondence has been established. This method is not, however, usually so useful as the dual of it which has been applied by Fink[†] and by Snyder[‡] to the classification of sextics. Consider two simple curves on a ruled surface such that each is cut by an arbitrary generator only once. To each point of one curve corresponds one point of the other lying on the same generator. Conversely, if a one to one correspondence is set up between the points of two curves, then the locus of the lines joining corresponding points is a ruled surface. This method has the advantage that it leads at once to the parametric equations of the surface.

6. Ruled surfaces known to belong to a given line complex may be studied by means of a curve theory if we apply a contact transformation which transforms the lines of the complex into the points of space. This is the method used by Wiman[§] in his classification of sextics and is the one which will chiefly be followed in the present investigation.

Notation and Theorems.

7. I shall use the following notation:

R_n = Scroll of degree n.
p = genus of a scroll.
P_i = point, i-fold on the surface.
g_i = i-fold generator.
$(2g_2)$ = Double torsal generator.[‖] See Par. 66.
$(3g_2)$ = A double torsal and a consecutive double generator. See Par. 67.
d_i = i-fold directrix.
$(d_i + jg_1)$ = i-fold directrix with which j generators coincide.
$(d + jg_1 + kg_2)$ = i-fold directrix with which j simple and k double generators coincide.

$(\delta_{k,l} + jg_1)$ = kl-fold contact directrix, *i. e.* one such that of the kl generators passing through a point, k lie in each of l planes through the directrix. In addition j generators coincide with the directrix.

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[*] Schwarz, Über die gradlinigen Flächen fünften Grades," Crelle's Journal, Vol. 67.

[†] Fink, "Über windschiefe Flächen," etc. Diss., Tübingen, 1886.

[‡] Snyder, "Classification of Sextic Scrolls," etc. *American Journal of Mathematics*, Vol. XXV, pp. 59–84, 85–96, 261–268. Vol. XXVII, pp. 77–102, 173–188.

[§] Wiman, "Regelytorna af Sjette Graden," Diss., Lund, 1892.

[‖] Wiman's "Singulär dubbelgeneratrix," l. b. i. d., p. 27.

$K_n =$ cone of order n.
$C^i_m =$ curve of order m which is i-fold on the surface.
$p' =$ genus of a curve.
$P'_i =$ point, i-fold on a curve.

8. The point of highest multiplicity that exists in general on a ruled surface is a P_3. If we consider a P_i equivalent to $\frac{i(i-1)(i-2)}{6} P_3$, then the number t, of P_3 on an R_n having a $(d_i + jg)$ is* $t = 1/6\,(n-i-j-2)\,[(n-i-j-1)(n+2i+2j-6)-6p]$.

9. The maximum genus of the nodal curve is

$$p' = 1/2\,(n-i-j-2)(n-i-j-3) + p\,(n-i-j-2)\dagger$$

10. Denoting by m' the number of intersections of a C^a_m with a $(d_i + jg)$ of an R_n and by b the number of intersections of the curve with an arbitrary generator, then

$$a(m-m') = b(n-i-j),$$

since each member of this equation equals the number of intersections of the curve with the $n-i-j$ generators in an arbitrary plane through the directrix.

11. It is also readily seen that for the entire nodal curve (including the directrix and multiple generators)

$$\sum \frac{ma(a-1)}{2} = \frac{(n-1)(n-2)}{2} - p$$

$$\sum (a-1)\,b = n-2$$

CLASSIFICATION OF SEPTIC SCROLLS HAVING A RECTILINEAR DIRECTRIX.

I. Scrolls Belonging to Linear Congruences.

a. Scrolls Belonging to General Linear Congruences.

12. The sum of the multiplicities i_1 and i_2 of the two rectilinear directrices of the scroll must equal n, the degree of the surface. Hence if we take $x = y = 0$ to be one directrix and $z = w = 0$ to be the other, then the equation of the surface must be homogeneous of degree i_1 in x and y, and of degree i_2 in z and w. Putting $\frac{x}{y} = \xi$, $\frac{z}{w} = \eta$ and regarding ξ, η as rectangular coordinates in a plane, we obtain the equation of a curve which is the transform, in the ξ, η plane, of

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* Wiman, loc. cit., p. 10.

† Wiman, loc. cit., p. 12.

the given surface.* It is, in general, of order n and has an i_1-fold point at infinity on the $\xi=0$ axis and an i_2-fold point at infinity on the $\eta=0$ axis. To the remaining multiple points of the curve correspond multiple generators of the scroll. It is, in fact, projective with the section of the surface by $y=w$.

13. For $n=7$ we may take for i_1 and i_2 the values 1 and 6, 2 and 5, or 3 and 4. We thus obtain the following R_7 having two rectilinear directrices.

$p=0$
1. d_1+d_6
2. $d_2+d_5+4g_2$
3. $d_3+d_4+6g_2$
4. $d_3+d_4+g_3+3g_2$
5. $d_3+d_4+2g_3$

$p=1$
6. $d_2+d_5+3g_2$
7. $d_3+d_4+5g_2$
8. $d_3+d_4+g_3+2g_2$

$p=2$
9. $d_2+d_5+2g_2$
10. $d_3+d_4+4g_2$
11. $d_3+d_4+g_3+g_2$

$p=3$
12. $d_2+d_5+g_2$
13. $d_3+d_4+3g_2$
14. $d_3+d_4+g_3$

$p=4$
15. d_2+d_5
16. $d_3+d_4+2g_2$

$p=5$
17. $d_3+d_4+g_2$

$p=6$
18. d_3+d_4

14. Particular cases of consecutive, tacnodal, oscnodal, etc., multiple generators may be obtained from corresponding particular cases of the transformed curve. I shall not usually call attention to these obvious particularizations.

b. Scrolls Belonging to Special Linear Congruences.

15. The equations of any special linear congruence may, by a suitable choice of coordinates, be written $p_{12}=0$, $p_{14}=p_{23}$. The equations of a scroll belonging to this congruence are:

$$x=a(u) \qquad z=c(u)+va(u)$$
$$y=b(u) \qquad w=d(u)+vb(u)$$

from which $\dfrac{x}{y}=\dfrac{a}{b}$; $\dfrac{xw-yz}{y^2}=\dfrac{ad-bc}{b^2}$. Hence the (x, y, z, w) equation of

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[*] For this transformation I am indebted to Professor Snyder.

the surface is* $\sum_{0}^{i}{}_{r}(xw - yz)^{r}.\ q_{n-2r}(x, y)$ where n is the order of the surface, $i \leqq \frac{n}{2}$ and $q_{n-2r}(x, y)$ is a binary quantic in x and y of degree $n - 2r$.

16. Putting $\frac{x}{y} = \xi$ and $\frac{xw - yz}{y^2} = \eta$ we obtain, as transform of the surface, a curve projective with the section of the surface by $y = w$. This curve is, in general, of order n. It has an $(n - i)$-fold and a consecutive i-fold point at infinity on the $\xi = 0$ axis. If, however, this point counts for more than $\frac{(n-i)(n-i-1)}{2} + \frac{i(i-1)}{2}$ double points, then multiple generators will coincide with the directrix. For $n = 7$, $i = 1, 2, 3$ we obtain:

$p = 0$

1. $(\delta_{1,1} + 5g_1)$
2. $(\delta_{2,1} + 3g_1) + 4g_2$
3. $(\delta_{2,1} + g_1 + g_2) + 3g_2$
4. $(\delta_{3,1} + g_1) + 6g_2$
5. $(\delta_{3,1} + g_1) + g_3 + 3g_2$
6. $(\delta_{3,1} + g_1) + 2g_3$

$p = 1$

7. $(\delta_{2,1} + 3g_1) + 3g_2$
8. $(\delta_{2,1} + g_1 + g_2) + 2g_2$
9. $(\delta_{3,1} + g_1) + 5g_2$
10. $(\delta_{3,1} + g_1) + g_3 + 2g_2$

$p = 2$

11. $(\delta_{2,1} + 3g_1) + 2g_2$
12. $(\delta_{2,1} + g_1 + g_2) + g_2$
13. $(\delta_{3,1} + g_1) + 4g_2$
14. $(\delta_{3,1} + g_1) + g_3 + g_2$

$p = 3$

15. $(\delta_{2,1} + 3g_1) + g_2$
16. $(\delta_{2,1} + g_1 + g_2)$
17. $(\delta_{3,1} + g_1) + 3g_2$
18. $(\delta_{3,1} + g_1) + g_3$

$p = 4$

19. $(\delta_{2,1} + 3g_1)$
20. $(\delta_{3,1} + g_1) + 2g_2$

$p = 5$

21. $(\delta_{3,1} + g_1) + g_2$

$p = 6$

22. $(\delta_{3,1} + g_1)$

II. Scrolls not Belonging to Linear Congruences.

Directrix a Simple Line on the Surface.

17. The nodal curve is limited very closely by the relations given in paragraphs 10 and 11. For the present case these reduce to $ma = 6b$, $\sum m\frac{a(a-1)}{2} = 15$, $\Sigma(a - 1)b = 5$. If $a = 6$, $m = 1$ and the R_7 belongs to a linear congruence, a can not equal 5 or 4. If $a = 3$ then $m = 2b$ hence every triple curve is of even order. Similarly, for a double curve, $m = 3b$.

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*Snyder, Bulletin Amer. Math. Soc. Vol. V, p. 351.

18. Since the surface is unicursal its parametric equations are:

$$x = a_6(u), \quad z = c_6(u) + v(\alpha u + \beta)$$
$$y = b_6(u), \quad w = d_6(u) + v(\gamma u + \delta)$$

where $a_6(u)$, $b_6(u)$, $c_6(u)$ and $d_6(u)$ are polynomials of sixth degree in u and $a_6(u)$ and $b_6(u)$ are relative prime.

19. If we form the function*

$$F_5(uu', u + u') = \frac{a_6(u)\, b_6(u') - b_6(u)\, a_6(u')}{u - u'},$$

then the number of components of the nodal curve and the genus of the components which are double curves are determined by the corresponding properties of $F_5 = 0$ considered as a curve whose current coordinates are $u u'$ and $u + u'$.

20. In the present case, the order of each double component of the nodal curve is thrice, and of each triple component is once, the order of the corresponding component of $F_5 = 0$.

21. The curve $F_5 = 0$ exhibits the following forms:

1. a proper C_5 of genus 6, 5, 4, 3, 2, 1, 0.
2. a C_1 and a C_4 of genus 3, 2, 1, 0.
3. a C_2 and a C_3 of genus 1, 0.
4. a C_1 and $2C_2$.
5. $3C_1$ and a C_2.

The forms $5C_1$ and $2C_1 + C_3$ do not exist provided $a_6(u)$ and $b_6(u)$ are relative prime.

a. *Triple Curves.*

22. A triple curve lying on an R_7 with a d_1 must be either a C_4^3 or a C_2^3.

To a C_4^3 on R_7 corresponds a quartic component of $F_5 = 0$. To the remaining linear component of $F_5 = 0$ corresponds a C_3^2 on R_7. This surface is readily obtained by Salmon's geometrical method (see p. 1). The scroll of bisecants from a unicursal C_4 to a straight line is, in general, an R_9 having the line for d_3. If however, the C_4 has a P_2' at which both tangents cut the straight line, then the scroll reduces to an R_7 having the line for d_1 since the P_2' will project from any point of the line into a tacnode. The R_7 has a P_4 at the P_2' of the C_4^3. The residual C_3^2 on the R_7 passes simply through that point. It also passes through each of the remaining points at which the tangent to the C_4^3 cuts d_1. The tangent to C_3^2 at each of these points also cuts d_1, since the six generators in the plane through the point and d_1 have 8 intersections at the point.

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* Cf. my article in this Journal, Vol. XXVIII, p. 43

23. To a C_2^3 on the R_7 corresponds a quadratic component of $F_5 = 0$. The R_7 can not have $2C_2^3$ for the three generators through a point of one C_2 would have to cut the other C_2 in two points in the plane through the given point and the directrix. The residual nodal curve is, therefore, C_9^2 or $C_6^2 + C_3^2$ or $3C_3^2$. The R_7 has a P_4 on the C_2^3 for the plane of the C_2 contains a g which cuts the C_2 in two points, fourfold in the curve of section. One is the point at which the plane touches the surface; the other is a P_4 on the surface. The residual nodal curve must have a P'_3 at the P_4 since the total nodal curve must have a P'_6 there. The residual curve also has a P'_3 at each point from which a tangent to C_2^3 cuts the directrix, for the six generators in the plane through the directrix and either of these points must consist of 3 torsals meeting at the point of tangency. When the double curve has a component C_3^2, two of the points at which the tangent to the C_3^2 cuts the directrix are these two points, the other two are points at which the C_3 cuts the remaining double curve.

24. We have, therefore, the following types:

1. $d_1 + C_4^3 + C_3^2$. R_7 has a P_4 at the P'_2 of the C_4. The C_3 is gauche. It passes through the P_4 and cuts C_4 in four other points at which the tangents to both curves cut d_1.

2. $d_1 + C_2^3 + C_9^2$. The C_9 has a P'_3 at the P_4 and a P'_3 at each point of the C_2^3 at which the tangent to C_2^3 cuts d_1. At each of these points the three tangents to C_9 cut d_1. p' for C_9 is in general 1, but may reduce to o by the appearance of a P'_2 at which both tangents cut d_1.*

3. $d_1 + C_2^3 + C_6^2 + C_3^2$. The C_3 goes through the P_4 and the points at which the tangents to C_2 cut d_1. The C_6 has a P'_2 at each of these three points and cuts the C_3 in two other points.

4. $d_1 + C_2^3 + 3C_3^2$. Each C_3 goes through the P_4 and the points from which the tangents to C_2 cut d_1. Each cuts each of the other C_3 again.

b. No Triple Curves.

25. The scroll determined by equations I, paragraph 18, may be made, by changes in the expressions for z and w only, to have for its point of highest multi-

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* In general, both tangents at a discrete P'_2 on a double curve must cut the directrix. Such points are the intersections of torsal generators having a common torsal plane.

plicity a P_5, a P_4 or a P_3 hence among each of these three kinds of scrolls are types determined by 1, 2, 3, 4, 5. No. 34.

Fivefold Point.

26. The double curve has a P'_{10} at the P_5 and therefore lies on a K_5. The curve $F_5 = 0$ considered above is projective with an arbitrary plane section of this K_5. Each C^2_m has a point of multiplicity $\frac{2}{3}m$ at the P_5. On each multiple generator of the K_5 lies P'_2 of the double curve. Hence we have:

1. $d_1 + C^2_{15}$. The C_{15} has a P'_{10} at the P_5. $p' = 6, 5, 4, 3, 2, 1, 0$ according as it has 0, 1, 2, 3, 4, 5, 6, P'_2.

2. $d_1 + C^2_3 + C^2_{12}$. The C_3 has a P'_2 and the C_{12} a P'_8 at P_5. They meet in four other points. p' for the C_{12} is 3, 2, 1, 0 according as it has 0, 1, 2, 3, P'_2.

3. $d_1 + C^2_6 + C^2_9$. The C_6 has a P'_4 and the C_9 a P'_6 at P_5. They meet in six other points. p' for C^2_9 is 1 or 0.

4. $d_1 + C^2_3 + 2C^2_6$. The C_3 has a P'_2 and each C_6 a P'_4 at P_5. The C_3 meets each C_6 in $2P'_1$ and the $2C_6$ meet in $4P'_1$.*

5. $d_1 + 3C^2_3 + C^2_6$. Each C_3 has a P'_2 and the C_6 a P'_4 at P_5. Each C_3 meets each of the other C_3 once and the C_6 twice.

Fourfold Point.

27. The double curve has a P'_3 at each of the $6P_3$ and a P'_6 at the P_4. In the plane through the P_4 and the d_1 lie $2g_1$ not passing through the P_4. When the double curve is composite, there are different types according as the point of intersection of these generators lies on one or another component.† The only apparent double points of the nodal curve from the P_4 are the four in the plane through the P_4 and d_1.

1. $d_1 + C^2_{15}$. The C_{15} has a P'_6 at the P_4 and a P'_3 at each of the $6P_3$. p' equals 6, 5, 4, 3, 2, 1, or 0 according to the number of its P'_2.

2. $d_1 + C^2_3 + C^2_{12}$. The C_3 passes through the P_4 and $4P_3$. The C_{12} has a P'_5 at the P_4, P'_3 at $2P_3$ and P'_2 at $4P_3$. It meets C_3 in $4P'_1$. p' for C_{12} is 3, 2, 1, 0.

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* The six generators in an arbitrary plane through the directrix are the sides of a Pascal hexagon. The vertices of the hexagon are the six points in the plane on one C^2_6, the conic on which they lie being the section, by the plane, of the K_2 with vertex at P_4 which contains this C^2_6. The vertices of the Steiner triangles are on the other C^2_6 and the points on the Pascal line are on the C^2_3.

† The existence of these distinct cases may be verified readily by constructing the R_7 so that the parameters of these generators satisfy first one and then another component of $F_5 = 0$.

3. $d_1 + C_3^2 + C_{12}^2$. The C_3 has a P'_2 at P_4 and passes through $2P_3$. The C_{12} has a P'_4 at P_4, P'_3 at $4P_3$ and P'_2 at $2P_3$. It meets C_3 in $4P'_1$. p' for C_{12} is 3, 2, 1 or 0.

4. $d_1 + C_6^2 + C_9^2$. The C_6 has a P'_3 at the P_4 and a P'_1 at each P_3. The C_9 has a P'_3 at the P_4 and a P'_2 at each P_3. It meets C_6 in $6P'$. p' for C_9 is 1 or 0.

5. $d_1 + C_6^2 + C_9^2$. The C_6 has a P'_2 at the P_4, a P'_3 at one P_3 and P'_1 at the remaining P_3. The C_9 has a P'_4 at the P_4 and P'_2 at $5P_3$. It meets C_6 in $6P'_1$. p' for C_9 is 1 or 0.

6. $d_1 + C_3^2 + 2C_6^2$. The C_3 passes through P_4 and through $4P_3$ and meets each C_6 in two other points. One C_6 has a P'_3 at the P_4, a P'_2 at one P_3, one P'_1 at $4P_3$. The other C_6 has a P'_2 at the P_4, a P'_3 at one P_3 and P'_1 at $5P_3$. It meets the other C_6 in $4P'_1$.

7. $d_1 + C_3^2 + 2C_6^2$. The C_3 as in type 6. One C_6 has a P'_3 at the P_4 and P'_1 at $6P_3$. The other C_6 has P'_2 at the P_4 and at $2P'_3$. It has P'_1 at the other $4P_3$ and meets the first C_6 in four other P'_1.

8. $d_1 + C_3^2 + 2C_6^2$. The C_3 has a P'_2 at the P_4, P'_1 at $2P_3$ and cuts each C_6 in two other P'_1. Each C_6 has a P'_2 at the P_4. One C_6 has a P'_3 at one P_3 and P'_1 at $5P_3$. The other C_6 has P'_2 at $3P_3$ and P'_1 at $2P_3$. They meet in four other P'_1.

9. $d_1 + 3C_3^2 + C_6^2$. Each C_3 has P'_1 at the P_4 and at $4P_3$. Each meets the C_6 in $2P'_1$ and each of the other C_3 in $1P'_1$. The C_6 has a P'_3 at the P_4 and a P'_1 at each P_3.

10. $d_1 + 3C_3^2 + C_6^2$. One C_3 has a P'_2 at P_4 and P'_1 at $2P_3$. The other $2C_3$ are as in type 9. The C_6 has a P'_2 at P_4, a P'_3 at one P_3 and P'_1 at the other $5P_3$.

Ten Threefold Points.

28. The R_7 cannot have a plane C_3^2 for the P'_2 on the C_3 would be at least a P_4 on the R_7. When any component of the nodal curve has a P'_1 at a P_3 then one generator through that point cuts it in one more point than an arbitrary generator does and conversely. Hence every C_3^2 has P'_1 at $6P_3$ since the R_7 has $6g_1$ in common with the R_4 of bisecants from the directrix to C_3. Similarly no C_6^2 can lie on a quadric for the total intersection of the quadric and R_7 would be of higher degree than 14.

1. $d_1 + C_{15}^2$. The C_{15} has a P'_3 at each P_3. $p' = 6, 5, 4, 3, 2, 1$ or 0.

2. $d_1 + C_3^2 + C_{12}^2$. The C_3 has P'_1 at $6P_3$. The C_{12} has P'_2 at these $6P_3$ and P'_3 at the other $4P_3$. It meets C_3 in $4P'_1$. p' for C_{12} is 3, 2, 1, or 0.

3. $d_1 + C_6^2 + C_9^2$. The C_6 has a P_3' at $1P_8$ and a P_1' at each of the other $9P_8$. The C_9 has P_2' at these $9P_8$ and meets the C_6 in six other P_1'. p' for C_9 is 1 or 0.

4. $d_1 + C_3^2 + 2C_6^2$. The C_3 goes through $6P_8$ and meets each C_6 in two other P_1'. One C_6 has a P_2' at one P_8 and a P_1' at each of the other $9P_8$. The other C_6 has P_2' at $3P_8$ and P_1' at six other P_8. It meets the other C_6 in four other P_1'.

5. $d_1 + 3C_3^2 + C_6^2$. Each C_3 goes through $6P_8$ and meets each of the other C_3 in one other P_1' and the C_6 in two other P_1'. The C_6 has a P_3' at $1P_8$ and a P_1' at each of the other $9P_8$.

Digression on a Point-Line Contact Transformation.

29. Of the infinite number of transformations that transform the lines of a special linear complex into the points of space, only one need be considered. For any such transformation is equivalent to a projection, followed by the transformation mentioned below, and that followed by a point transformation. This transformation has already been fully discussed by Wiman.*

30. The polar planes for a point P with respect to a pencil of quadric surfaces form a pencil of planes having a line L for axis. Let the pencil of quadrics be so chosen that one member consists of two planes intersecting in a line M. Then the lines L determined by all the points of space cut M. A one to one correspondence is thus established between the points of space and the lines of a special linear complex whose axis is M.

31. To the points of a line of the complex correspond, by the theory of pole and polar, the lines of the complex through the corresponding point. To an R belonging to the complex corresponds a C in space. To any curve C' on R corresponds a ruled surface containing C. In particular, to the double curve of R corresponds the ruled surface formed by the complex lines which cut C twice.

Two successive applications of the transformation obviously produce identity.

32. Let the equations of the pencil of quadrics be:

$$x^2 + y^2 + \lambda\,(y^2 + z^2 + w^2) = 0.$$

To the point $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$, then, corresponds the line determined by any two of the equations:

$$x\bar{x} + y\bar{y} = 0 \tag{1}$$

$$y\bar{y} + z\bar{z} + w\bar{w} = 0 \tag{2}$$

$$x\bar{x} - z\bar{z} - w\bar{w} = 0 \tag{3}$$

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* Regelytorna of Sjette Graden, Lund, Diss., 1892, p. 17.

33. The R corresponding to the intersection of two surfaces $\phi_1(\bar{x}, \bar{y}, \bar{z}, \bar{w}) = 0$ and $\phi_2(\bar{x}, \bar{y}, \bar{z}, \bar{w}) = 0$ is found by eleminating $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ between $\phi_1 = 0$, $\phi_2 = 0$ and any two of equations (1), (2), and (3). The surface corresponding to the curve $\bar{x} = f_1(u)$, $\bar{y} = f_2(u)$, $\bar{z} = f_3(u)$, $\bar{w} = f_4(u)$ is found by substituting these values in (1), (2) and (3) and eliminating u. The curve corresponding to a given ruled surface is found in a similar manner.

34. The fundamental points of the correspondence, *i. e.*, those which make any two of equations (1), (2) and (3) identical, are $\bar{x} = \bar{z} = \bar{w} = 0$, $\bar{y} = \bar{z} = \bar{w} = 0$ and the points of $\bar{x} = \bar{y} = 0$. The order of the scroll corresponding to given curve is obviously diminished by unity for every time the curve goes through a fundamental point.

35. For convenience the points $\bar{x} = \bar{z} = \bar{w} = 0$ and $\bar{y} = \bar{z} = \bar{w} = 0$ only will be referred to as fundamental points. To the point $\bar{x} = \bar{z} = \bar{w} = 0$ on a curve corresponds, obviously, a line in $y = 0$. If the direction of the curve at this point is given by $d\bar{x}$, $d\bar{z}$, $d\bar{w}$ then the line in $x = 0$ is determined by $xd\bar{x} - zd\bar{z} - wd\bar{w}$. Similarly for curves through $\bar{y} = \bar{z} = \bar{w}$. To the ∞^2 lines in $x = 0$ and $y = 0$ thus correspond the ∞^2 directions through the fundamental points.

36. By equations (1) and (3), it is seen that to all the points in $\bar{x} = 0$ on a line through $\bar{x} = \bar{z} = 0$ (except $\bar{x} = \bar{z} = \bar{w} = 0$ itself and the point in $\bar{y} = 0$) correspond the same line $y = 0$, $z\bar{z} + w\bar{w} = 0$. Multiple generators on R are thus accounted for either by multiple points on C or by points on lines of the complex through the fundamental points, according as the generator does not or does pass through a singular point.

37. Let the tangent to C at a point of intersection with $\bar{x} = \bar{y} = 0$ lie in the plane $\bar{x}d\bar{y} - \bar{y}d\bar{x} = 0$. The corresponding line is $xd\bar{x} + yd\bar{y} + 0$, $z\bar{z} + w\bar{w} = 0$. It obviously cuts $z = w = 0$. The number of intersections of R with $z = w = 0$ at points other than fundamental points equals the number of intersections of C with $\bar{x} = \bar{y} = 0$.

38. To the points of C in any plane $z + \lambda\bar{w} = 0$ (except the fundamental points) correspond generators of R through $(0, 0, 1, \lambda)$. To the points common to all the planes correspond lines through all the points; *i. e.*, the directrix counts as many times as a generator as C intersects $\bar{z} = \bar{w} = 0$ in points other than fundamental points.

39. To a C_m cutting $\bar{x} = \bar{y} = 0$ α times, having a P'_β at $\bar{x} = \bar{z} = \bar{w} = 0$ and a P'_γ at $\bar{y} = \bar{z} = \bar{w} = 0$ corresponds an $R_{2m-\alpha-\beta-\gamma}$ having $x = y = 0$ for an

$(m-\beta-\gamma)$-fold line, having a $P_{m-\alpha-\beta}$ at $y=z=w=0$ and a $P_{n-\alpha-\gamma}$ at $x=z=w=0$. Similarly to an R_n having $x=y=0$ for α-fold line, having a P_β at $x=z=w=0$ and a P_γ at $y=z=w=0$ corresponds a $C_{2m-\alpha-\beta-\gamma}$ cutting $\bar{x}=\bar{y}=0$, $n-\beta-\gamma$ times, having a $P'_{n-\alpha-\beta}$ at $\bar{y}=\bar{z}=\bar{w}=0$ and a $P'_{n-\alpha-\gamma}$ at $\bar{x}=\bar{z}=\bar{w}=0$.

40. To the pencil of complex lines in an arbitrary plane corresponds a conic, cutting $\bar{x}=\bar{y}=0$ and passing through both fundamental points. To the points of the plane correspond the complex lines cutting the conic. We may, therefore, consider this conic as the transform of the plane. Since each tangent plane to R contains a generator, the corresponding conic cuts C. To the planes of the double developable of R (*i. e.*, the developable enveloped by planes containing two generators of R) correspond the conics which cut C twice. But these conics are in one to one correspondence with the lines of the complex $p_{34}=0$ which cut C twice. Hence the double developable is in one to one correspondence with the scroll of bisecants from $\bar{z}=\bar{w}=0$ to C.

41. To the points of an arbitrary straight line corresponds a quadric surface containing $x=y=0$ and the fundamental points. To an R of the complex which has this line for a second rectilinear directrix corresponds, therefore, a C lying on this quadric surface. In particular when the quadric contains $z=w=0$ the corresponding line is $\bar{x}=\bar{y}=0$. To a C lying on such an R_2 corresponds an R belonging to a special linear congruence.

42. To any line passing through a fundamental point corresponds a plane pencil passing through the other fundamental point. To an R of the complex having this line for a second directrix corresponds a C in the plane of the corresponding pencil.

43. To a point of C at which the tangent cuts $\bar{x}=\bar{y}=0$ corresponds a torsal generator of R with its pinch-point not on the directrix. To a point of C at which the tangent cuts $\bar{z}=\bar{w}=0$ corresponds a torsal with its pinch-point on the directrix.

44. It is occasionally more convenient to use another transformation. A brief mention of some of its properties is therefore appended. It is projectively equivalent to Euler's contact transformation.* It was also discussed by Wiman.† When it is necessary to distinguish between these two transformations the one above will be referred to as transformation I and the following as transformation II.

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* Lie-Scheffers Berührungstransformationen, 1896, Vol. I, p. 647.

† Wiman, loc. cit. p. 23.

45. Let the equation of the pencil of quadrics be:

$$2xy + (2yw + z^2) = 0$$

The transformation is determined by the equations:

$$x\bar{y} + y\bar{x} = 0$$
$$y\bar{w} + z\bar{z} + w\bar{y} = 0$$

46. The singular points are those on $x = y = 0$ and on $z = w = 0$. The degree of the surface corresponding to C is decreased by unity for each of its intersections with either of these lines. If C passes through the point of intersection of the lines, the degree is reduced only by unity unless C touches $\bar{y} = 0$ at the point, in which case the degree is reduced by two.

47. To points of C in $\bar{y} = 0$ not on the fundamental lines correspond generators coincident with $y = z = 0$. To points on $\bar{y} = \bar{z} = 0$ correspond generators in $y = 0$ and to points on $\bar{x} = \bar{y} = 0$ correspond generators through $x = y = z = 0$ not, in general, in $y = 0$.

48. Hence to a C_n having α points on $\bar{x} = \bar{y} = 0$ and β on $\bar{y} = \bar{z} = 0$ corresponds an $R_{2n-\alpha-\beta}$ having $x = y = 0$ for $d_{n-\beta}$ and $y = z = 0$ for $g_{n-\alpha-\beta}$. Similarly to an R_n having $x = y = 0$ for d_α and $y = z = 0$ for g_β corresponds a $C_{n\,-}$ having $\alpha - \beta$ points on $\bar{x} = \bar{y} = 0$ and $n - \alpha - \beta$ points on $\bar{y} = \bar{z} = 0$.

Directrix a Double Line on the Surface.

$$p = 0$$

Fivefold Point.

49. Taking the P_5 and any other point of the nodal curve for fundamental points in depiction I, the R_7 is transformed into a C_5 with a P'_3 at one fundamental point and not passing through the other. The scroll of bisecants is an R_{13} whose genus is, in general, 3 but may reduce to 0. To determine how it may break up, take the second fundamental point temporarily on the C_5. The C_5 then depicts into an R_6 with a P_4. From the known properties of the nodal curve of such an R_6,* it is readily seen that the R_{13} can break up only into an R_6 with a d_1 and an R_7 with a d_2.

50. The line joining the fundamental points may cut the C_5 in one point not a fundamental point. The directrix then counts once as a directrix and once as a generator $(d_1 + g_1)$.

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* See Wiman, loc. cit. p. 26. Snyder, American Journal of Math., Vol. 27, p. 77.

51. When the R_7 has a g_2, it must pass through the P_5. The fundamental point not on C_5 must then lie on the scroll of bisecants of the C_5. If the R_7 has $2g_2$ then the fundamental point is on the intersection of two generators of the scroll of bisecants (see paragraph 51).

52. When the fundamental point not on C_5 lies on the line joining the points of tangency of two tangents which are coplanar with $\bar{x} = \bar{y} = 0$, then the R_7 has a singularity first noticed by Wiman* which I have called a double torsal generator since it is formed by two coincident torsals having a common torsal plane. The pinch points are, in general, distinct. The line then counts for $2g_2$ as a component of the nodal curve but for only $1g_2$ as a component of the double developable. In the case of the surface we are now considering, however, the pinch points must coincide. Such a $(2g_2)$ counts for $2g_2$ as a component both of the nodal curve and of the double developable.

53. The line on which the fundamental point lies is itself a $(2g_2)$ of the R_{13}. When the fundamental point lies at the pinch point of either torsal of this $(2g_2)$ the multiple line of the R_7 counts, in general, for $3g_2$ as a component of the nodal curve and for $2g_2$ as a component of the double developable. I shall denote it by $(3g_2)$. It should be noticed in the present case, however, that the two pinch points on the $(3g_2)$ coincide and it therefore counts for $3g_2$ as a component of the double developable.

54. Hence we have:

1. $d_2 + C^2_{14}$; $(d_1 + g_1) + C^2_{14}$. The C_{14} has a P'_{10} at the P_5. $p' = 3, 2, 1, 0$ according to the number of discrete P'_2. It meets the directrix four times.

2. $d_2 + 2C^2_7$; $(d_1 + g_1) + 2C^2_7$. Each C_7 has a P'_5 at the P_5 and cuts the other C_7 in $4P'_1$. Each C_7 meets the directrix twice.

3. $d_2 + g_2 + C^2_{18}$. The C_{18} has a P'_9 at the P_5. It meets the g_2 once again. $p' = 3, 2, 1, 0$. It meets the directrix three times.

4. $d_2 + g_2 + C^2_6 + C^2_7$. The C_6 has a P'_4 at the P_5, meets the g_2 once again and the C_7 four times again. The C_7 has a P'_5 at the P_5.

5. $d_2 + 2g_2 + C^2_{12}$. The C_{12} has a P'_8 at the P_5 and meets each g_2 again. $p' = 3, 2, 1, 0$.

6. $d_2 + 2g_2 + C^2_5 + C^2_7$. The C_5 has a P'_3 at P_5 and cuts each g_2 once and the C_7 four times again. The C_7 has a P'_5 at the P_5.

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* Wiman, loc. cit. p. 27.

7. $d_2 + 2g_2 + 2C_6^2$. Each C_6 has a P_4' at the P_5 and meets the other C_6 in four other points. Each cuts $1g_2$ in a P_1'.

8. $d_2 + (2g_2) + C_{12}^2$. The C_{12} has at the P_5 a P_8' with two branches touching the $(2g_2)$. $p' = 2, 1, 0$.

9. $d_2 + (2g_2) + 2C_6^2$. Each C_6 has at the P_5 a P_4' with one branch touching the $(2g_2)$. They meet in $3P_1'$.

10. $d_2 + (3g_2) + C_{11}^2$. The C_{11} has at the P_5 a P_7' with one branch touching the $(3g_2)$. It cuts the latter again with its tangent in the plane through the directrix. $p' = 2, 1, 0$.

11. $d_2 + (3g_2) + C_6^2 + C_5^2$. The C_5 has a P_3' at the P_5, touches the torsal plane of the $(3g_2)$ at another point on the latter and meets the C_6 in $3P_1'$. The C_6 as in (9).

Fourfold Point.

55. Taking the P_4 and a P_3 for fundamental points,* the R_7 is transformed into a C_5 with a P_2' at one fundamental point and a P_1' at the other. The scroll of bisecants is an R_{15} with a d_5. It may be seen from the parametric equations of the R_7 that the R_{15} is of genus 3, 2, 1, 0 and that it can break up only into two unicursal components each having the C_5 for double curve. Since no scroll of degree six or less can have such a C_5^2, the R_{15} can break up only into an R_7 and an R_8. One component has a P_4 and the other a P_3 at the P_2'. If the R_7 had a P_4 there, a line could be drawn through the point meeting the R_7 in eight points. Hence the R_7 has a P_3 there and the R_8 a P_4. The R_7 has $2g_2$ and the R_8 has $4g_2$ one of which passes through the P_2'.

56. The line joining the fundamental points may cut the C_5 in another point giving rise to a (d_1+g_1) or the fundamental point not at the P_2' may be on one, or at the intersection of two trisecants to the C_5 cutting $\bar{x} = \bar{y} = 0$, in which case the R_7 has $1g_2$, or $2g_2$, through the P_4. In particular the two trisecants may be consecutive the R_7 then has a $(2g_2)$ with pinch-points coincident at the P_4.

57. The R_7 may have through the P_4 a $(2g_2)$ the pinch-points of which are not coincident. This happens when the tangent to the C_5, at the fundamental point, cuts the curve at another point and is coplanar with the tangent at the point of intersection and $\bar{x} = \bar{y} = 0$.

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*In some particular cases this cannot be done; but the R_7 may still be transformed into such a C_5 by taking the fundamental points elsewhere and following the contact transformation by a point transformation which leaves the linear complex $p_{12} = 0$ invariant.

58. Hence we have:

1. $d_2 + C_{14}^2$; $(d_1 + g_1) + C_{14}^2$. The C_{14} has a P_6' at the P_4 and $6P_8'$. $p' = 3, 2, 1, 0$.

2. $d_2 + 2R_7^2$; $(d_1 + g_1) + 2R_7^2$. Each C_7 has a P_3' at the P_4, P_2' at $3P_8$ and P_1' at the other $3P_8$. They meet in $4P_1'$.

3. $d_2 + g_2 + C_{13}^2$. The C_{13} has a P_5' at the P_4, a P_2' and a P_1' on the g_2 and $5P_2'$. $p' = 3, 2, 1, 0$. The g_2 goes through the P_4 and a P_8.

4. $d_2 + g_2 + C_7^2 + C_6^2$. The C_7 has a P_3' at the P_4, P_2' at $3P_8$ and P_1' at the remaining $2P_8$ not on g_2. It cuts the g_2 in a P_1'. The C_6 has P_2' at the P_4 and at 3 discrete P_8 and P_1' at the other two. They meet in $4P_1'$. The g_2 goes through P_4 and a P_8.

5. $d_2 + g_2 + C_7^2 + C_6^2$. This differs from (4) in that C_6 and C_7 each pass simply through the P_8 on the g_2 and the C_7 cuts it in another P_1'.

6. $d_2 + 2g_2 + C_{12}^2$. The C_{12} has a P_4' at the P_4, P_3 at $4P_8$ and a P_2' and a P_1' on each g_2. $p' = 3, 2, 1, 0$. Each g_2 goes through the P_4 and one P_8.

7. $d_2 + 2g_2 + 2C_6^2$. Each C_6 has a P_2' at the P_4, a P_2' on one g_2 and a P_1' on the other g_2. Each has P_2' at two of the remaining P_8 (which must lie on the same g_1) and passes simply through the other $2P_8$. They meet in $4P_1'$. Each g_2 goes through the P_4 and a P_8.

8. $d_2 + (2g_2) + C_{12}^2$. The C_{12} has a P_4' at the P_4 with two branches touching the $(2g_2)$. It touches itself again on the $(2g_2)$ and has P_3 at the 4 discrete P_8. $p' = 2, 1, 0$. The $(2g_2)$ has its pinch-points coincident at the P_4 and passes through $2P_8$ which are consecutive.

9. $d_2 + (2g_2) + 2C_6^2$. Each C_6 has a P_2' at the P_4 with one branch touching the $(2g_2)$. They touch again on the $(2g_2)$. Each has P_2' at two discrete P_8 and P_1' at the other two. The $(2g_2)$ as in (8).

10. $d_2 + (2g_2) + C_{12}^2$. One pinch-point of the $(2g_2)$ is at P_4, the other at a P_8. The C_{12} has a P_4' at the P_4, passes through the other pinch-point of the $(2g_2)$ and touches its torsal plane in two other points on the generator. The C_{12} has a P_3' at each of the 5 discrete P_8. $p' = 2, 1, 0$.

11. $d_2 + (2g_2) + 2C_6^2$. The $(2g_2)$ as in (10). Each C_6 has a P_2' at the P_4, touches the torsal plane of the $(2g_2)$ at another point of it and meets the other C_6 in $3P_1'$. One C_6 has P_2' at $2P_8$ and P_1' at the other four. The other C_6 has P_2' at $3P_8$ and P_1' at the other two discrete P_8.

59. The R_7 may have a g_2 not passing through P_4. It then transforms into a C_5 with $2P_2'$. Such a C_5 may be transformed into an R_6 with a d_1. From

the known properties of the nodal curve of such an R_6 it follows that the R_{14} of bisecants of the C_5 is of genus 3, 2, 1, 0 and may break up into an R_8 and R_6 or into $2R_7$.

60. The fundamental point not at a P'_2 of the C_5 may lie on a trisecant of the C_5 cutting $\bar{x}=\bar{y}=0$, in which case the R_7 has a g_2 through the P_4 in addition to the other. It may also lie at the vertex of a K_2 containing the C_5. According as $\bar{x}=\bar{y}=0$ cuts the K_2 in distinct or consecutive points, the R_7 has $2g_2$ or a $(2g_2)$ with coincident pinch points through the P_4. The R_7 may also have a $(2g_2)$ with distinct pinch-points passing through the P_4 since the configuration on the C_5, as mentioned above, to produce this singularity may exist in this case also.

61. The multiple generator not passing through the P_4 may be a $(2g_2)$ with pinch-points necessarily not coincident.* Both tangents at the corresponding P'_2 of the C_5 then cut $\bar{x}=\bar{y}=0$. The scroll of bisecants is an R_{13} of genus 2, 1, 0. It may break up into an R_7 and an R_6. It is seen as before that the original R_7 may have through P_4 a g_2 or a $(2g_2)$ with distinct pinch-points. When the R_7 has $2g_2$, or a $(2g_2)$ with coincident pinch-points, through the P_4, it has no discrete P_3. It may, however, be transformed into one of the C_5 mentioned above by taking a multiple generator through P_4 for $y=z=0$ in transformation II.

62. There exists no R_7 with a P_4 and $2g_2$ neither of which passes through the P_4. For, suppose such an R_7 to exist. Take one of these g_2 for $y=z=0$ in transformation II. A C_5 with a quadrisecant and a P'_2 not on the quadrisecant would then be obtained.

12. $d_2+g_2+C^2_{18}$. The g_2 goes through $3P_3$. The C_{18} has a P'_6 at the P_4, $3P'_2$ and a P'_1 on the g_2 and P'_3 at the other $3P_3$. $p'=3, 2, 1, 0$.

13. $d_2+g_2+C^2_7+C^2_6$. The g_2 goes through $3P_3$. The C_7 has a P'_3 at the P_4, a P'_2 on the g_2 and at each of 2 discrete P_3 and P'_1 at the other $3P_3$. The C_6 has a P'_3 at the P_4. It meets the C_7 twice on the g_2 and cuts the g_2 again. It has a P'_2 at one, and P'_1 at the other two discrete P_3. It meets the C_7 in four other P'_1.

14. $d_2+2g_2+C^2_{12}$. One g_2 goes through the P_4 and a P_3; the other, through $3P_3$. The C_{12} has a P'_5 at the P_4. It has P'_2 at each P_3 on the $2g_2$ and cuts each g_2 in a P'_1. It has P'_3 at the remaining $2P_3$. $p'=3, 2, 1, 0$.

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* They may, however, become consecutive. The corresponding P'_2 of the C_5 is then a cusp.

15. $d_2 + 2g_2 + 2C_6^2$. One g_2 goes through the P_4 and a P_3, the other, through, $3P_3$. One C_6 has a P_2' at the P_4, $2P_1'$ on the g_2 through the P_4, a P_2' and $2P_1'$ on the other g_2, a P_2' at one discrete P_3 and a P_1' at the other. The other C_6 has a P_3' at the P_4. It meets the other C_6 once on the g_2 through the P_4 and twice on the other g_2 and cuts the latter again. It has a P_2' at one discrete P_3 and a P_1' at the other, It meets the other C_6 in four other P_1'.

16. $d_2 + 2g_2 + C_7^2 + C_5^2$. One g_2 goes through the P_4 and a P_3; the other, through $3P_3$. The C_7 has a P_3 at the P_4, and a P_2' on the g_2 through the P_4. It has a P_3' and $2P_1'$ on the other g_2. It has a P_2' at one discrete P_3 and a P_1' at the other. The C_5 has a P_2' at the P_4 and a P_1' on the g_2 through the P_4. It meets the C_7 twice on the other g_2 and cuts this g_2 again. It has a P_2' at one discrete P_3 and a P_1' at the other. It meets the C_7 in four other P_1'.

17. $d_2 + 3g_2 + C_{11}^2$. $2g_2$ go through the P_4 and a P_3, the other goes through $3P_3$. The C_{11} has a P_4' at the P_4, P_2' at the P_3 on the $3g_2$ and a P_3' at the discrete P_3. It has a P_1' on each g_2. $p' = 3, 2, 1, 0$.

18. $d_2 + 3g_2 + C_6^2 + C_5^2$. The $3g_2$ as in (17). The C_6 and C_5 each have a P_2' at P_4, a P_2' on one g_2 through the P_4 and a P_1' on the other. The C_6 meets the third g_2 in a P_2' and $2P_1'$ and has a P_2' at the discrete P_3. The C_5 meets the C_6^2 twice on the g_2 through $3P_3$, cuts this g_2 again, and has a P_1' at the discrete P_3. It meets the C_6 in four other P_1'.

19. $d_2 + (2g_2) + g_2 + C_{11}^2$. The $(2g_2)$ has its pinch-points coincident at the P_4 and goes through $2P_3$. The g_2 goes through $3P_3$. The C_{11} has a P_4' with two branches touching the $(2g_2)$ at P_4. It touches itself again on the $(2g_2)$, cuts the g_2 in $3P_2'$ and a P_1' and has a P_3' at the discrete P_3. $p' = 2, 1, 0$.

20. $d_2 + (2g_2) + g_2 + C_6^2 + C_5^2$. The $(2g_2)$ and g_2 as in (19). The C_6 and C_5 each have a P_2' with one branch touching the $(2g_2)$ at P_4. They touch at another point of the $(2g_2)$, intersect twice on the g_2 and in three other P_1'. The C_6 meets the g_2 again in a P_2'. The C_5 meets it in a P_1'. The C_6 has a P_2' and the C_5 a P_1' at the discrete P_3.

21. $d_2 + (2g_2) + g_2 + C_{11}^2$. One pinch-point of the $(2g_2)$ is at P_4, the other at a P_3. The g_2 goes through $3P_3$. The C_{11} has a P_4' at P_4. It has a P_1' at the P_3 on the $(2g_2)$, and touches its torsal plane in two other points of it. It has P_2' at the $3P_3$ on g_2 and P_3' at the discrete $2P_3$. $p' = 2, 1, 0$.

22. $d_2 + (2g_2) + g_2 + C_6^2 + C_5^2$. The $(2g_2)$ and g_2 as in (21). The C_6 and C_5 each have a P_2' at P_4, touch the torsal plane of the $(2g_2)$ at another point of it, have a P_2' at one discrete P_3 and a P_1' at the other and meet in three discrete P_1'.

The C_6 has a P'_1 at the P_3 on the $(2g_2)$, a P'_2 and $2P'_1$ on the g_2. The C_5 has $3P'_1$ on the g_2.

23. $d_2 + (2g_2) + C^2_{12}$. The $(2g_2)$ passes through $4P_3$ (one at each pinch-point, two, consecutive, at the intersection with the g_1 in the plane through the $(2g_2)$ and d_1). The C_{12} has a P'_6 at P_4 and P'_3 at the two discrete P_3. It touches itself on the $(2g_2)$, goes through the pinch-points, and touches the torsal plane of the $(2g_2)$, at two other points of it. $p' = 2, 1, 0$.

24. $d_2 + (2g_2) + 2C^2_6$. The $(2g_2)$ as in (23). Each C_6 has a P'_3 at the P_4, touches the other C_6 on the $(2g_2)$ and meets the $(2g_2)$ twice again. Each has a P'_2 at one discrete P_3 and a P'_1 at the other. They meet in three other P'_1.

25. $d_2 + (2g_2) + g_2 + C^2_{11}$. The $(2g_2)$ as in (23). The g_2 goes through the P_4 and a P_3. The C_{11} has a P'_5 at the P_4 and a P'_2 and a P'_1 on the g_2. It has a P'_3 at the discrete P_3 and two consecutive P'_2 and $4P'_1$ on the $(2g_2)$. $p' = 2, 1, 0$.

26. $d_2 + (2g_2) + g_2 + C^2_6 + C^2_5$. The $(2g_2)$ and g_2 as in (25). The C_6 has a P'_3 at the P_4 and a P'_2 on the g_2. It has a P'_1 at the discrete P_3. The C_5 has a P'_2 at the P_4, meets the g_2 again and has a P'_2 at the discrete P_3. The C_5 and C_6 touch on the $(2g_2)$ and each meets it in two more points. They meet in three other P'_1.

27. $d_2 + (2g_2) + g_2 + C^2_6 + C^2_5$. This differs from (26) in that the C_6 has a P'_1 and not a P'_2 on the g_2 and a P'_2 and not a P'_1 at the discrete P_3 and the C_5 has $2P'_1$ on the g_2 and a P'_1 at the discrete P_3.

28. $d_2 + (2g_2) + 2g_2 + C^2_{10}$. The $(2g_2)$ as in (23). Each g_2 goes through the P_4 and a P_3. The C_{10} has a P'_4 at the P_4 and a P'_2 and a P'_1 on each g_2. It has two consecutive P'_2 and $4P'_1$ on the $(2g_2)$. $p' = 2, 1, 0$.

29. $d_2 + (2g_2) + 2g_2 + 2C^2_5$. The $(2g_2)$ and $2g_2$ as in (28). Each C_5 has a P'_2 at the P_4, a P'_2 on one g_2 and a P'_1 on the other. They touch on the $(2g_2)$ and each cuts the $(2g_2)$ in two more points. They meet in three more P'_1.

30. $d_2 + 2(2g_2) + C^2_{10}$. One $(2g_2)$ as in (23). The other has its pinch-points coincident at the P_4 and passes through two P_3. The C_{10} has a P'_4 with two branches touching the $(2g_2)$ at the P_4. It has two consecutive P'_2 on each g_2 and cuts the $(2g_2)$ not passing through the P_4 in four other points. $p' = 1, 0$.

31. $d_2 + 2(2g_2) + 2C^2_5$. The $2(2g_2)$ as in (30.) Each C_5 has a P'_2 at the P_4 with one branch touching the $(2g_2)$. They touch on each $(2g_2)$. Each meets the $(2g_2)$ not passing through the P_4 twice again.

32. $d_2 + 2\,(2g_2) + C^2_{10}$. One $(2g_2)$ has one pinch-point at the P_4, the other at a P_3. The other $(2g_2)$ passes through $4P_3$. The C_{10} has a P'_4 at the P_4, a P'_1

at each of the other three pinch points, and a P'_3 at the discrete P_3. It meets each $(2g_2)$ twice again. $p' = 1, 0$.

33. $d_2 + 2\,(2g_2) + 2C^2_5$. The $2\,(2g_2)$ as in (32). Each C_5 has a P'_2 at the P_4. One has a P_2 at the discrete P_3. The other P'_1 at the discrete P_3 and at the P_3 on the $(2g_2)$ through the P_4. They touch on the other $(2g_2)$ and meet in two discrete P'_1. Each C_5 meets each $(2g_2)$ again.

Ten Threefold Points.

63. From the parametric equations of the surface it is readily seen that the C^2_{14} of the R_7 is of genus, 3, 2, 1, 0 and that it may break up into two unicursal components each cut by an arbitrary generator twice. At each P_3 one component has a P'_2 and the other a P'_1 but the second component has through the point a trisecant cutting the directrix. Conversely, on each trisecant of each component which cuts the curve in $3P'_1$ not on the directrix, lies a P_3 of the R_7 at which the other component has a P'_2. Hence, for each component, the sum of the number of these trisecants and of the number of P'_2 is ten. The C^2_{14} can not, therefore, break up into $C^2_9 + C^2_5$ for the maximum of that sum for a C_5 is 8.* Neither can it break up into $C^2_8 + C^2_6$ for the C_6 must have $2P'_2$ and the C_8, $8P'_2$ but such a C_8 must lie on a quadric. The only possible components are, therefore, $2C^2_7$ each having $5P'_2$ and five trisecants of the kind mentioned.

64. It is readily seen that the R_7 may have a g_2 and seven discrete P_3 and that, as before, the double curve may break up. One component has a P'_2 and $2P'_1$ on the g_2, the other, $3P'_1$. For each component, therefore, the sum of the P'_2 and of the trisecants is eight. If, therefore, the C^2_{13} broke up into $C^2_8 + C^2_5$, the C_5 could have no P'_2 and the C_8 would have to have $8P'_2$ and lie on a quadric. Hence the C^2_{13} decomposes into $C^2_7 + C^2_6$.

65. The multiple generator may be a $(2g_2)$ passing through $4P_3$. When the C^2_{12} is composite, the $(2g_2)$ counts for 2 consecutive trisecants on each component. Hence the C^2_{12} can break up only into $2C^2_6$.

66. From these considerations we obtain:

1. $d_2 + C^2_{14}$; $(d_1 + g_1) + C^2_{14}$. The C_{14} has a P'_3 at each P_3. $p' = 3, 2, 1, 0$.

2. $d_2 + 2C^2_7$; $(d_1 + g_1) + 2C^2_7$. Each C_7 has P'_2 at $5P_3$ and P'_1 at the other $5P_3$. They meet in $4P'_1$. Both are unicursal.

3. $d_2 + g_2 + C^2_{13}$. The C_{13} has $3P'_2$ and a P'_1 on the g_2 and P'_3 at the seven discrete P_3. $p' = 3, 2, 1, 0$.

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* See paragraph 4.

4. $d_2 + g_2 + C_7^2 + C_6^2$. The C_7 has a P_2' and $2P_1'$ on the g_2, P_2' at four discrete P_3 and P_1' at the other three. The C_6 has $3P_1$ on the g_2, P_2' at two discrete P_3 and P_1' at the other five. It meets the C_7 in four other P_1'. Both are unicursal.

5. $d_2 + (2g_2) + C_{12}^2$. The C_{12} touches itself on the $(2g_2)$ and meets the $(2g_2)$ four times again. It has P_3' at the six discrete P_3. $p' = 2, 1, 0$.

6. $d_2 + (2g_2) + 2C_6^2$. Each C_6 touches the other on the $(2g_2)$ and meets the $(2g_2)$ twice again. Each has P_2' at three discrete P_3 and P_1' at the other three. They meet in three other P_1'. Both are unicursal.

67. When the R_7 has $2g_2$, take one of them for $y = z = 0$ in transformation II. The R_7 then transforms into a C_5 having a P_2' and having $\bar{y} = \bar{z} = 0$ for trisecant. We have seen* that the scroll of bisecants to such a C_5 is an R_{15} which may break up into an R_8 and an R_7. The components of the C_{12}^2 are $C_7^2 + C_5^2$ or $2C_6^2$ according as $\bar{y} = \bar{z} = 0$ is a g_1 of the R_8 and a g_2 of the R_7 or vice versa.

68. When one of the multiple generators is a $(2g_2)$ both tangents to the C_5 at the P_2' cut $\bar{x} = \bar{y} = 0$. The scroll of bisecants is an R_{14} of genus 2, 1, 0 which may break up into two R_7. $\bar{y} = \bar{z} = 0$ is a g_2 on one component R_7 and a g_1 on the other. When both multiple generators are $(2g_2)$, the C_5 also touches $\bar{y} = 0$ at two of its intersections with $\bar{y} = \bar{z} = 0$. When the scroll of bisecants is composite, $\bar{y} = \bar{z} = 0$ is a multiple generator on each component.

69. When the R_7 has $3g_2$ the C_5 has $2P_2'$. Both tangents at one P_2' may cut $\bar{x} = \bar{y} = 0$ in which case the R_7 has $2g_2 + (2g_2)$.

7. $d_2 + 2g_2 + C_{12}^2$. The C_{12} has $3P_2'$ and a P_1' on each g_2 and P_3' at the four discrete P_3. $p' = 3, 2, 1, 0$.

8. $d_2 + 2g_2 + 2C_6^2$. Each C_6 has a P_2' and $2P_1'$ on one g_2 and $3P_1'$ on the other. Each has P_2' at two discrete P_3 and P_1' at the other two. They meet in four other P_1'. Both are unicursal.

9. $d_2 + 2g_2 + C_7^2 + C_5^2$. The C_7 has a P_2' and $2P_1'$ on each g_2, a P_2' at three discrete P_3 and a P_1' at the other. The C_5 has $3P_1'$ on each g_2, a P_2' at one discrete P_3 and P_1' at the other three. The C_5 and C_7 meet in four other P_1'. Both are unicursal.

10. $d_2 + (2g_2) + g_2 + C_{11}^2$. The C_{11} has a $3P_2'$ and a P_1' on the g_2. It touches itself on the $(2g_2)$ and meets the $(2g_2)$ four times again. It has P_3' at the three discrete P_3. $p' = 2, 1, 0$.

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*See paragraph 55.

11. $d_2 + (2g_2) + g_2 + C_6^2 + C_5^2$. The C_6 has a P_2' and $2P_1'$ on the g_2, $3P_1'$ on the $(2g_2)$, P_2' at two discrete P_3 and a P_1' at the other. The C_5 cuts each multiple generator thrice, has a P_2' at one discrete P_3 and a P_1' at the other two. It meets the C_6 in three other P_1'. Both are unicursal.

12. $d_2 + 2\,(2g_2) + C_{10}^2$. The C_{10} has two consecutive P_2' and $4P_1'$ on each $(2g_2)$, and P_3' at the two discrete P_3. $p' = 1$, or 0.

13. $d_2 + 2(2g_2) + 2C_5^2$. Each C_5 touches the other on each $(2g_2)$ and meets each $(2g_2)$ twice again. Each has a P_2' at one discrete P_3, and a P_1' at the other. They meet in two more P_1'. Both are unicursal.

14. $d_2 + 3g_2 + C_{11}^2$. The C_{11} has $3P_2'$ and a P_1' on each g_2 and a P_3' at the discrete P_3. $p' = 3, 2, 1, 0$.

15. $d_2 + 3g_2 + C_6^2 + C_5^2$. The C_6 has a P_2' and $2P_1'$ on each of $2g_2$ and $3P_1'$ on the other. It has a P_2' at the discrete P_3. The C_5 has a P_2' and $2P_1'$ on $1g_2$ and $3P_1'$ on each of the other two. It has a P_1' at the discrete P_3 and cuts the C_6 in four more P_1'. Both curves are unicursal.

16. $d_2 + (2g_2) + 2g_2 + C_{10}^2$. The C_{10} has $3P_2'$ and a P_1' on each g_2 and $2P_2'$ and $4P_1'$ on the $(2g_2)$. $p' = 2, 1, 0$.

17. $d_2 + (2g_2) + 2g_2 + 2C_5^2$. Each C_5 has a P_2' and $2P_1'$ on one g_2 and $3P_1'$ on the other g_2. They touch on the $(2g_2)$ and each meets the $(2g_2)$ twice again. They meet in three other P_1'. Each is unicursal.

$p = 1$.

Fourfold point.

70. The R_7 is transformed by I into a C_5 of genus 1, having a P_2'. The scroll of bisecants is an R_{14}. The fundamental point not at P_2' may be on a trisecant of the C_5 cutting $\bar{x} = \bar{y} = 0$. It may also be at the vertex of a K_2 which contains the C_5 and cuts $\bar{x} = \bar{y} = 0$ in two distinct or coincident points.

71. The R_7 cannot have a g_2 not passing through the P_4 for then it would transform by II into a C_5 of genus 1 having a quadrisecant.

The double curve cannot break up. For, the cone having its vertex at P_4 and containing one component would have to be unicursal. The component itself would, therefore, be unicursal. This is impossible.

1. $d_2 + C_{13}^2$. The C_{13} has a P_6' at the P_4 and P_3' at the $3P_3$. p' is, at most, 6.

2. $d_2 + g_2 + C_{12}^2$. The g_2 goes through the P_4. The C_{12} has a P_5' at the

P_4 and meets the g_2 again in a P_2' and a P_1'. It has P_3' at the two discrete P_3. Its maximum genus is 6.

3. $d_2 + (2g_2) + C_{11}^2$. The $(2g_2)$ has its pinch-points coincident at the P_4. The C_{11} has a P_4' with two branches touching the $(2g_2)$ at the P_4, touches itself again on the $(2g_2)$ and has a P_3' at the discrete P_3. Its maximum genus is 5.

4. $d_2 + 2g_2 + C_{11}^2$. Each g_2 goes through the P_4 and a P_3. The C_{11} has a P_4' at the P_4, a P_2' and a P_1' on each g_2 and a P_3' at the discrete P_3. Its maximum genus is 6.

Seven threefold points.

72. The R_7 is transformed by I into a C_6 of genus 1 with $2P_2'$. When the C_6 has a third P_2' the R_7 has a g_2 or a $(2g_2)$. When the R_7 has $2g_2$ it is transformed by II into such a C_5 as was found for the transform in the case of a P_4. One, but not both, multiple generators may become double torsal. The double curve is not decomposable.

1. $d_3 + C_{18}^2$. The C_{18} has P_3' at the $7P_3$. Its maximum genus is 6.

2. $d_2 + g_2 + C_{12}^2$. The C_{12} has $3P_2'$ and a P_1' on the g_2 and P_3' at the 4 discrete P_3. Its maximum genus is 6.

3. $d_2 + (2g_2) + C_{11}^2$. The C_{11} has 2 consecutive P_2' and $4P_1'$ on the $(2g_2)$ and P_3' at the 3 discrete P_3. Its maximum genus is 5.

4. $d_2 + 2g_2 + C_{11}^2$. The C_{11} has $3P_2'$ and a P_1' on each g_2 and a P_3' at the discrete P_3. Its maximum genus is 6.

5. $d_3 + (2g_2)' + g_2 + C_{10}^2$. The C_{10} has $3P_2'$ and a P_1' on the g_2 and 2 consecutive P_2' and $4P_1'$ on the $(2g_2)$. Its maximum genus is 5.

$$p = 2.$$

73. The R_7 has $4P_3$. The double curve is not decomposable.

1. $d_2 + C_{12}^2$. The C_{12} has P_3' at the $4P_3$. Its maximum genus is 9.

2. $d_2 + g_2 + C_{11}^2$. The C_{11} has $3P_2'$ and a P_1' on the g_2 and a P_3' at the discrete P_3. Its maximum genus is 9.

3. $d_2 + (2g_2) + C_{10}^2$. The C_{10} has 2 consecutive P_2' and $4P_1'$ on the $(2g_2)$. Its maximum genus is 8.

$$p = 3.$$

74. The R_7 has one P_3. The double curve is not decomposable.

1. $d_2 + C_{11}^2$. The C_{11} has a P_3' at P_3. Its maximum genus is 12.

Directrix a threefold line on the surface.

$p = 0.$

Fourfold point.

75. The R_7 is transformed by I into a C_4 of the second kind passing through one fundamental point. Transform the C_4 by II into an R_5. From the known properties of the nodal curve of such an R_5,[*] we find that the R_9 of bisecants of the C_4 may break up into an R_6 and an R_3 or into $3R_3$. $\bar{z} = \bar{w} = 0$ meets the C_4 in 0, 1, 2 points besides the fundamental point on the curve.

76. When the fundamental point not on the C_4 is on the scroll of bisecants the R_7 has a g_2 through the P_4. $\bar{z} = \bar{w} = 0$ meets the C_4 in 0, 1 points besides the fundamental point on the C_4. It cannot meet it in two more points for then it would meet 10 generators of the scroll of bisecants.

77. Similarly, this fundamental point may be at the intersection of two generators of the scroll of bisecants, giving rise to $2g_2$ through the P_4. When the C_4 has two tangents coplanar with the directrix the fundamental point may lie on a $(2g_2)$ of the scroll of bisecants (in case the scroll of bisecants is composite these will be coincident simple torsals of two components) giving rise to a $(2g_2)$ of the R_7 with pinch-points coincident at the P_4. In particular, the fundamental point may lie at a pinch-point of the $(2g_2)$ of the scroll of bisecants. The R_7 then has a $(3g_2)$ through the P_4.

1. $d_3 + C_{12}^2$; $(d_2 + g_1) + C_{12}^2$; $(d_1 + 2g_1) + C_{12}^2$. The C_{12} has a P_6' at the P_4 and P_3' at the $3P_8$. $p' = 1, 0$.

2. $d_3 + C_8^2 + C_4^2$; $(d_2 + g_1) + C_8^2 + C_4^2$; $(d_1 + 2g_1) + C_8^2 + C_4^2$. The C_8 has a P_4' at the P_4 and P_2' at the $3P_8$. The C_4 has a P_2' at the P_4 and P_1' at the $3P_8$. It meets the C_8 in $2P_1'$.

3. $d_3 + 2C_4^2$; $(d_2 + g_1) + 3C_4^2$; $(d_1 + 2g_1) + 3C_4^2$. Each C_4 has a P_2' at the P_4 and P_1' at the $3P_8$. Each two C_4 meet in another P_1'.

4. $d_3 + g_2 + C_{11}^2$; $(d_2 + g_1) + g_2 + C_{11}^2$. The g_2 goes through the P_4. The C_{11} has a P_5' at the P_4 and P_8' at the $3P_8$. It meets the g_2 again in a P_1'. $p' = 1, 0$.

5. $d_3 + g_2 + C_7^2 + C_4^2$; $(d_2 + g_1) + g_2 + C_7^2 + C_4^2$. The g_2 goes through the P_4. The C_7 has a P_3' at the P_4, P_2' at the $3P_8$, a P_1' on the g_2 and $2P_1'$ on the C_4. The C_4 has a P_2' at the P_4 and P_1' at the $3P_8$.

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[*] See Snyder, On the forms of quintic scrolls, Bulletin of Am. Math. Society, 2d Series, Vol. VIII, p. 298.

6. $d_3 + g_2 + C_8^2 + C_3^2$; $(d_2 + g_1) + C_8^2 + C_3^2$. The g_2 goes through the P_4. The C_8 has a P'_4 at the P_4 and P'_2 at the $3P_3$. The C_3 is gauche, has P'_1 at the P_4 and at the $3P_3$. It meets the g_2 again in a P'_1 and the C_8 in $2P'_1$.

7. $d_3 + g_2 + 2C_4^2 + C_3^2$; $(d_2 + g_1) + g_2 + 2C_4^2 + C_3^2$. The g_2 goes through the P_4. Each C_4 has a P'_2 at the P_4 and P'_1 at the $3P_3$. Each meets the other C_4 and the C_3 in another P'_1. The C_3 is gauche, has P'_1 at the P_4 and at the $3P_3$ and meets the g_2 again in a P'_1.

8. $d_3 + 2g_2 + C_{10}^2$; $(d_2 + g_1) + 2g_2 + C_{10}^2$. The $2g_2$ pass through the P_4. The C_{10} has a P'_4 at the P_4 and P'_3 at the $3P_3$. $p' = 1, 0$.

9. $d_3 + 2g_2 + C_6^2 + C_4^2$; $(d_2 + g_1) + 2g_2 + C_6^2 + C_4^2$. The $2g_2$ pass through the P_4. The C_6 has P'_2 at the P_4 and at the $3P_3$ and a P'_1 on each g_2. The C_4 has a P'_2 at the P_4 and P'_1 at the $3P_3$. The C_6 and C_4 meet in $2P'_1$.

10. $d_3 + 2g_2 + C_8^2 + C_2^2$. The $2g_2$ pass through the P_4. The C_8 has a P'_4 at the P_4 and P'_2 at the $3P_3$. The C_2 has P'_1 at the $3P_3$, a P'_1 on each g_2 and meets the C_8 in $2P'_1$. It does not pass through the P_4.

11. $d_3 + 2g_2 + 2C_4^2 + C_2^2$. The $2g_2$ pass through the P_4. Each C_4 has a P'_2 at the P_4 and P'_3 at the $3P_3$. Each meets the other C_4 and the C_2 in a P'_1. The C_2 has a P'_1 on each g_2 and P'_1 at the $3P_3$.

12. $d_3 + (2g_2) + C_{10}^2$; $(d_2 + g_1) + (2g_2) + C_{10}^2$. The $(2g_2)$ goes through the P_4. The C_{10} has P'_3 at the $3P_3$. At the P_4 it has a P'_4 with two branches touching the $(2g_2)$. $p' = 0$.

13. $d_3 + (2g_2) + C_7^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + C_7^2 + C_3^2$. The $(2g_2)$ goes through the P_4. The C_7 has, at the P_4, a P'_3 with one branch touching the $(2g_2)$. It has a P'_2 at each P_3. The C_3 is gauche, touches the $(2g_2)$ at the P_4, has P'_1 at each P_3 and meets the C_7 again in a P'_1.

14. $d_3 + (2g_2) + C_4^2 + 2C_3^2$; $(d_2 + g_1) + (2g_2) + C_4^2 + 2C_3^2$. The $(2g_2)$ goes through the P_4. The C_4 has a P'_2 at the P_4 and P'_1 at the $3P_3$. Each C_3 is gauche, touches the $(2g_2)$ at the P_4, has P'_1 at the $3P_3$ and meets the C_4 again in a P'_1.

15. $d_3 + (3g_2) + C_9^2$; $(d_2 + g_1) + (3g_2) + C_9^2$. The $(3g_2)$ goes through the P_4. The C_9 has P'_3 at the P_4 and at the $3P_3$. It touches the $(3g_2)$ at the P_4 and touches its torsal plane at another point of it. $p' = 0$.

16. $d_3 + (3g_2) + C_6^2 + C_3^2$; $(d_2 + g_1) + (3g_2) + C_6^2 + C_3^2$. The $(3g_2)$ goes through the P_4. The C_6 has P'_2 at the P_4 and at the $3P_3$. It touches the torsal plane of the $(3g_2)$ at another point of it. The C_3 is gauche, touches the $(3g_2)$ at the P_4 and has P'_1 at the $3P_3$. It meets the C_6 in a P'_1.

17. $d_2 + (3g_2) + C_7^2 + C_2^2$. The $(3g_2)$ goes through the P_4. The C_7 has, at the P_4, a P_3' with one branch touching the $(3g_2)$. It has a P_2' at each P_3 and meets the C_2 in a P_1'. The C_2 touches the torsal plane of the $(3g_2)$ at a point of it and has P_1' at the $3P_3$.

18. $d_3 + (3g_2) + C_4^2 + C_3^2 + C_2^2$. The $(3g_2)$ goes through the P_4. The C_4 has a P_2' at the P_4, P_1' at the $3P_3$ and meets the C_3 and C_2 each again in a P_1'. The C_3 is gauche, touches the $(3g_2)$ at the P_4 and has P_1' at the $3P_3$. The C_2 touches the torsal plane of the $(3g_2)$ at a point of it and has P_1' at the $3P_3$.

78. When the scroll of bisecants of the C_4 into which the R_7 is transformed has a component R_3, the d_2 of this R_3 may be taken for $\bar{z} = \bar{w} = 0$. The fundamental point on the C_4 then lies on a trisecant of the C_4 cutting $\bar{x} = \bar{y} = 0$. It therefore gives rise on the R_7 to a g_2 not passing through the P_4. The fundamental point not on the C_4 is either at the intersection of $2g_1$ or at a pinch-point of a $(2g_2)$ of the scroll of bisecants and gives rise correspondingly to two intersecting g_2 or to a $(3g_2)$. The C_4 has $2P_1'$ on $\bar{z} = \bar{w} = 0$. One of them is a fundamental point; the other transforms into a g_1 coincident with the directrix. The two remaining generators which pass through a point of the directrix also lie in a plane through the directrix since they are the transforms of points lying on a complex line cutting $\bar{z} = \bar{w} = 0$. The directrix is, therefore, a contact directrix* with which a generator coincides, $(\delta_{2,1} + g_1)$. An R_7 is the simplest surface that can have this singularity apart from the restricted case of cubics and quintics contained in a special linear congruence.

79. In general when C lies on an R having $\bar{x} = \bar{y} = 0$ and $\bar{z} = \bar{w} = 0$ for directrices and when the C has j points other than fundamental points on $\bar{z} = \bar{w} = 0$ and has k points on each of the l generators of the R which lie in an arbitrary plane through $\bar{z} = \bar{w} = 0$, then the C transforms into an R which has $x = y = 0$ for $(\delta_{k,1} + jg_1)$.

19. $(\delta_{2,1} + g_1) + 3g_2 + C_3^2$. $2g_2$ pass through the P_4; the other, through $2P_3$. The C_3 has a P_4' at the P_4 and a P_2' at each P_3 of which two are on a g_2 and the other necessarily on the directrix. It meets the directrix in two more P_1'.

20. $(\delta_{2,1} + g_1) + 3g_2 + 2C_4^2$. $2g_2$ pass through the P_4; the other through $2P_3$. Each C_4 has a P_2' at the P_4 a P_1' at each P_3 (of which one is on the directrix) and cuts the directrix once again.

21. $(\delta_{2,1} + g_1) + (3g_2) + g_2 + C_7^2$. The $(3g_2)$ passes through the P_4; the g_2 through $2P_3$. The C_7 has at the P_4 a P_3' with one branch touching the $(3g_2)$. It

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* Wiman, loc. cit. p. 43.

has P'_2 at the $3P_3$ (of which one is on the directrix) and meets the directrix in two other points.

22. $(\delta_{2,1} + g_1) + (3g_2) + g_2 + C^2_4 + C^2_3$. The $(3g_2)$ and g_2 as in (21). The C_4 has a P'_2 at the P_4, P'_1 at the $3P_3$ (of which one is on the directrix) and cuts the directrix again. The C_3 is gauche, touches the $(3g_2)$ at the P_4, has P'_1 at the $3P_3$ and meets the C_4 in another P'_1.

80. The remaining cases in which the R_7 has a g_2 not passing through the P_4 may be more conveniently transformed into C_4 having a P'_2. Take temporarily for fundamental points the P'_2 and a P'_1 and trapsform the C_4 into an R_5. From the know properties of the nodal curve of such an R_5,* we find that the R_3 of bisecants to the C_4 may break up into an R_5 and an R_3, or into an R_6 and a K_2, or into $2R_3$ and a K_2.

81. In addition to the fundamental point on the C_4, $\bar{z} = \bar{w} = 0$ may cut the C_4 in a P'_1 or at the P'_2. In the latter case, the R_7 has two generators with coincident torsals coinciding with the directrix. As the directrix now counts for four instead of for three double lines as a component of the nodal curve, I have denoted this singularity by $(d_1 + g_2)$. It should be noticed, however, that the singularity is analogous to a $(2g_2)$ and it differs from both the g_2 and $(2g_2)$ in that it may occur, as in this case, on a line which counts only once as a directrix.

22. The fundamental point not on the C_4 may lie on one or on two g_1 of the scroll of bisecants, giving rise to one g_2 or two g_2 through the P_4. It may also lie on a $(2g_2)$ or, in particular, at a pinch-point of a $(2g_2)$ of the scroll of bisecants, giving rise to a $(2g_2)$ or a $(3g_2)$ through the P_4 on the R_7.

23. $d_3 + g_2 + C^2_{11}$; $(d_2 + g_1) + g_2 + C^2_{11}$; $(d_1 + g_2) + C^2_{11}$. The C_{11} has a P'_6 at the P_4, P'_2 at the $2P_3$ on the g_2 and a P'_3 at the discrete P_3. $p' = 1, 0$.

24. $d_3 + g_2 + C^2_8 + C^2_3$; $(d_2 + g_1) + g_2 + C^2_8 + C^2_3$; $(d_1 + g_2) + C^2_8 + C^2_3$. The C_8 has a P'_4 at the P_4, P'_2 at the $3P_3$ and meets the C_3 in $2P'_1$. The C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and a P'_1 on the g_2.

25. $d_3 + g_2 + C^2_7 + C^2_4$; $(d_2 + g_1) + g_2 + C^2_7 + C^2_4$; $(d_1 + g_2) + C^2_7 + C^2_4$. The C_7 has a P'_4 at the P_4, a P'_2 at the discrete P_3 and $3P'_1$ on the g_2. The C_4 has a P'_2 at the P_4, a P'_1 at the discrete P_3, meets the C_7 in the $2P_3$ on the g_2 and in two other P'_1.

26. $d_3 + g_2 + 2C^2_4 + C^2_3$; $(d_2 + g_1) + g_2 + 2C^2_4 + 2C^2_3$; $(d_1 + g_2) + 2C^2_4 + C^2_3$. Each C_4 has a P'_2 at the P_4, P'_1 at the $3P_3$ and meets the other C_4 and the C_3

See Snyder, On the forms of quintic scrolls, Bulletin of the Amer. Math. Soc., 2d Series, Vol. 8, 293.

each again in a P_1'. The C_8 has a P_2' at the P_4, a P_1' at the discrete P_8 and a P_1' on the g_2.

27. $d_8 + 2g_2 + C_{10}^2$; $(d_2 + g_1) + 2g_2 + C_{10}^2$. One g_2 goes through the P_4; the other, through $2P_8$. The C_{10} has a P_5' at the P_4, a P_3' at the discrete P_8 and a $2P_2'$ and a P_1' on the g_2. $p' = 1, 0$.

28. $d_8 + 2g_2 + C_8^2 + C_2^2$. The $2g_2$ as in (27). The C_8 has a P_4' at the P_4 and P_2' at the $3P_8$. It meets the C_2 in $2P_1'$. The C_2 has P_1' at the P_4 and at the discrete P_8. It meets each g_2 once.

29. $d_8 + 2g_2 + C_7^2 + C_8^2$; $(d_2 + g_1) + 2g_2 + C_7^2 + C_8^2$. The $2g_2$ as in (27). The C_7 has a P_3' at the P_4 and a P_2' at the $3P_8$. It meets the g_2 through the P_4 in a P_1' and the C_8 in $2P_1'$. The C_8 has a P_2' at the P_4, a P_1' at the discrete P_8 and a P_1' on the g_2 not passing through the P_4. (This C_8 transforms into a K_2. The one in (30) transforms into an R_8.)

30. $d_8 + 2g_2 + C_7^2 + C_8^2$; $(d_2 + g_1) + 2g_2 + C_7^2 + C_8^2$. The $2g_2$ as in (27). The C_7 has a P_4' at the P_4, a P_2' at the discrete P_8 and $3P_1'$ on the g_2 not passing through the P_4. The C_8 has a P_1' at each multiple point, meets the C_7 once and the g_2 through the P_4 twice, again.

31. $d_8 + 2g_2 + C_6^2 + C_4^2$; $(d_2 + g_1) + 2g_2 + C_6^2 + C_4^2$. The $2g_2$ as in (27). The C_6 has a P_3' at the P_4 and a P_2' at the discrete P_8. It meets the g_2 through the P_4 in a P_1' and the other g_2 in $3P_1'$. The C_4 has a P_2' at the P_4 and P_1' at the $3P_8$. It meets the C_6 in $2P_1'$.

32. $d_8 + 2g_2 + 2C_4^2 + C_2^2$. The $2g_2$ as in (27). Each C_4 has a P_2' at the P_4, P_1' at the $3P_8$ and meets the other C_4 and the C_2 in another P_1'. The C_2 has a P_1' at the P_4 and at the discrete P_8.

33. $d_8 + 2g_2 + C_4^2 + 2C_8^2$; $(d_2 + g_1) + 2g_2 + C_4^2 + 2C_8^2$. The $2g_2$ as in (27). The C_4 has a P_2' at the P_4 and a P_1' at each P_8. One C_8 has P_1' at the P_4 and at the $3P_8$ and meets the g_2 through the P_4 again. The other C_8 has a P_2' at the P_4, a P_1' at the discrete P_8 and has a P_1' on the g_2 passing through $2P_8$. Each C_8 meets the C_4 and the other C_8 again in a P_1'.

34. $d_8 + 3g_2 + C_9^2$; $(d_2 + g_1) + 3g_2 + C_9^2$. $2g_2$ pass through the P_4; the other through $2P_8$. The C_9 has a P_4' at the P_4, a P_3' at the discrete P_8 and P_2' at the other $2P_8$. It meets each g_2 in a P_1'. $p' = 1, 0$.

35. $d_8 + 3g_2 + C_7^2 + C_2^2$. The $3g_2$ as in (34). The C_7 has a P_4' at the P_4, a P_2' at the discrete P_8 and $3P_1'$ on the g_2 not passing through the P_4. The C_2 has P_1' at the $3P_8$, meets the C_7 in $2P_1'$ and has a P_1' on each g_2 through the P_4.

36. $d_3 + 3g_2 + C_6^2 + C_3^2$. The $3g_2$ as in (34). The C_6 has P'_2 at the P_4 and at the $3P_3$, and meets each g_2 through the P_4 again. The C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and meets the C_6 in two other P'_1. It has a P'_1 on the g_2 passing through $2P_3$.

37. $d_3 + 3g_2 + C_5^2 + C_4^2$. The $3g_2$ as in (34). The C_5 has P'_2 at the P_4 and at the discrete P_3. It meets each g_2 through the P_4 in a P'_1 and the other g_2 in $3P'_1$. The C_4 has a P'_2 at the P_4, P'_1 at the $3P_3$ and two other P'_1 on the C_5.

38. $d_3 + 3g_2 + C_4^2 + C_3^2 + C_2^2$. The $3g_2$ as in (34). The C_4 has a P'_2 at the P_4 and P'_1 at the $3P_3$. The C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and a P'_1 on the g_2 through $2P_3$. The C_2 has P'_1 at the $3P_3$ and a P'_1 on each g_2 through the P_4. Each pair of the curves C_4, C_3 and C_2 have another P'_1 in common.

39. $d_3 + (2g_2) + g_2 + C_9^2$. The $(2g_2)$ has its pinch-points coincident at the P_4. The g_2 goes through $2P_3$. The C_9 has at the P_4 a P'_4 with two branches touching the $(2g_2)$. It has a P'_3 at the discrete P_3 and $2P'_2$ and a P'_1 on the g_2. $p' = 0$.

40. $d_3 + (2g_2) + g_2 + C_7^2 + C_2^2$. The $(2g_2)$ and the g_2 as in (39). The C_7 has at the P_4 a P'_3 with one branch touching the $(2g_2)$. It has P'_2 at the $3P_3$. The C_2 touches the $(2g_2)$ at the P_4, has a P'_1 at the discrete P_3 and meets the C_7 again. It has a P'_1 on the g_2.

41. $d_3 + (2g_2) + g_2 + C_6^2 + C_3^2$. The $(2g_2)$ and the g_2 as in (39). The C_6 has, at the P_4, a P'_3 with one branch touching the $(2g_2)$. It has a P'_2 at the discrete P_3 and $3P'_1$ on the g_2. The C_3 touches the $(2g_2)$ at the P_4, has P'_1 at the $3P_3$ and meets the C_6 again.

42. $d_3 + (2g_2) + g_2 + C_4^2 + C_3^2 + C_2^2$. The $(2g_2)$ and the g_2 as in (39). The C_4 has a P'_2 at the P_4 and P'_1 at the $3P_3$. The C_3 touches the $(2g_2)$ at the P_4 and has P'_1 at the $3P_3$. The C_2 touches the $(2g_2)$ at the P_4, has a P'_1 at the discrete P_3 and meets the g_2. The C_4 meets the C_3 and the C_2 each in another P'_1.

43. $d_3 + (2g_2) + g_2 + 3C_3^2$. The $(2g_2)$ and the g_2 as in (39). One C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and meets the g_2. Each of the other $2C_3$ touches the $(2g_2)$ at the P_4 and has P'_1 at the $3P_3$. The first C_3 meets each of the others again.

44. $d_3 + (3g_2) + g_2 + C_8^2$. The pinch-points on the $(3g_2)$ are coincident at the P_4. The g_2 goes through $2P_3$. The C_8 has, at the P_4, a P'_3 with one branch

touching the $(3g_2)$. It touches the torsal plane of the $(3g_2)$ at another point of it. The C_3 has a P'_3 at the discrete P_3 and $2P'_2$ and a P'_1 on the g_2. $p' = 0$.

45. $d_3 + (3g_2) + g_2 + C_6^2 + C_2^2$. The $(3g_2)$ and the g_2 as in (44). The C_6 has P'_2 at the P_4 and at the $3P'_3$ and touches the torsal plane of the $(3g_2)$ at another point of it. The C_2 touches the $(3g_2)$ at the P_4, has a P'_1 at the discrete P_3 and meets the g_2. It meets the C_6 again in a P'_1.

46. $d_3 + (3g_2) + g_2 + C_6^2 + C_3^2$. The $(3g_2)$ and the g_2 as in (44). The C_6 has at the P_4 a P'_3 with one branch touching $(3g_2)$. It has a P'_2 at the discrete P_3 and a $3P'_1$ on the g_2. The C_2 has a P'_1 at the discrete P_3, meets the C_6 in $2P'_1$ on the g_2 and in another P'_1 and touches the torsal plane of the $(3g_2)$ at a point of the $(3g_2)$.

47. $d_3 + (3g_2) + g_2 + C_5^2 + C_3^2$. The $(3g_2)$ and the g_2 as in (44). The C_5 has a P'_2 at the P_4 and touches the torsal plane of the $(3g_2)$ at another point of the latter. It has a P'_2 at the P_3 and $3P'_1$ on the g_2. The C_3 touches the $(3g_2)$ at P_4, has a P'_1 at the P_3 and meets the C_5 in $2P'_1$ on the g_2 and in another P'_1.

48. $d_3 + (3g_2) + g_2 + C_4^2 + 2C_2^2$. The $(3g_2)$ and the g_2 as in (44). The C_4 has a P'_2 at the P_4 and P'_1 at the $3P_3$. One C_2 touches the $(3g_2)$ at the P_4, has a P'_1 at the discrete P_3 and meets the g_2. The other has P'_1 at the $3P_3$ and touches the torsal plane of the $(3g_2)$ at a point of the latter. The C_4 meets each in another P'_1.

49. $d_3 + (3g_2) + g_2 + 2C_3^2 + C_2^2$. The $(3g_2)$ and the g_2 as in (44). One C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and meets the g_2. The other C_3 touches the $(3g_2)$ at the P_4 and has P'_1 at the $3P_3$. The C_2 has P'_1 at the $3P_3$ and touches the torsal plane of the $(3g_2)$ at a point of the latter. The first C_3 meets the second C_3 and the C_2 each in another P'_1.

83. When the R_7 has a $(2g_2)$ not passing through the P_4, it may be transformed by I into a C_4 with a P'_2 at which the tangent cuts $\bar{x} = \bar{y} = 0$. The scroll of bisecants is an R_7 of genus zero, which may break up into an R_5 and a K_2 or into an R_3 and $2K_2$.

84. In addition to the fundamental point on the C_4 the line $\bar{z} = \bar{w} = 0$ may cut the C_4 in a P'_1 or at the P'_2. In the latter case an arbitrary plane section of the corresponding R_7 has, at the trace of $x = y = 0$, a P'_3 at which two branches have contact of the second order. $x = y = 0$ counts for five as a component of the nodal curve. I shall denote the singularity by $[d_1 + (2g_2)]$.

85. As before, it is seen that the R_7 may have through the P_4 a g_2, $2g_2$, a $(2g_2)$, or a $(3g_2)$.

50. $d_3 + (2g_2) + C^2_{10}$; $(d_2 + g_1) + (2g_2) + C^2_{10}$; $[d_1 + (2g_2)] + C^2_{10}$. The $(2g_2)$ goes through $2P_3$. The C_{10} has a P'_6 at the P_4, a P'_3 at the discrete P_3 and four points on the $(2g_2)$. $p' = 0$.

51. $d_3 + (2g_2) + C^2_7 + C^2_3$; $(d_2 + g_1) + (2g_2) + C^2_7 + C^2_3$; $[d_1 + (2g_2)] + C^2_7 + C^2_3$. The $(2g_2)$ as in (50). The C_7 has a P'_4 at the P_4, a P'_2 at the discrete P_3 and $3P'_1$ on the $(2g_2)$. The C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and a P'_1 on the $(2g_2)$. It meets the C_7 again in a P'_1.

52. $d_3 + (2g_2) + C^2_4 + 2C^2_3$; $(d_2 + g_1) + (2g_2) + C^2_4 + 2C^2_3$; $[d_1 + (2g_2)] + C^2_4 + 2C^2_3$. The $(2g_2)$ as in (50). The C_4 has a P'_2 at the P_4 and a P'_1 at the $3P_3$. Each C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3, another P'_1 on the C_4 and a P'_1 on the $(2g_2)$.

53. $d_3 + (2g_2) + g_2 + C^2_9$; $(d_2 + g_1) + (2g_2) + g_2 + C^2_9$. The $(2g_2)$ goes through $2P_3$; the g_2, through the P_4. The C_9 has a P'_5 at the P_4 and a P'_3 at the discrete P_3. It meets the g_2 again and has $4P'_1$ on the $(2g_2)$. $p' = 0$.

54. $d_3 + (2g_2) + g_2 + C^2_7 + C^2_2$. The $(2g_2)$ and the g_2 as in (53). The C_7 has a P'_4 at the P_4, a P'_2 at the discrete P_3 and $3P'_1$ on the $(2g_2)$. The C_2 has P'_1 at the P_4 and at the discrete P_3, meets C_7 and the g_2 each again and has a P'_1 on the $(2g_2)$.

55. $d_3 + (2g_2) + g_2 + C^2_6 + C^2_3$; $(d_2 + g_1) + (2g_2) + g_2 + C^2_6 + C^2_3$. The $(2g_2)$ and the g_2 as in (53). The C_6 has a P'_3 at the P_4, a P'_2 at the discrete P_3 and P'_1 at the $2P_3$ on the $(2g_2)$. It meets the $(2g_2)$, the g_2 and the C_3 each again. The C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and a P'_1 on the $(2g_2)$.

56. $d_3 + (2g_2) + g_2 + C^2_4 + C^2_3 + C^2_2$. The $(2g_2)$ and the g_2 as in (53). The C_4 has a P'_2 at the P_4, P'_1 at the $3P_3$ and meets the C_3 and C_2 each again. The C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and a P'_1 on the $(2g_2)$. The C_2 has a P'_1 at the P_4 and at the discrete P_3, meets the g_2 again and meets the $(2g_2)$.

57. $d_3 + (2g_2) + g_2 + 3C^2_3$; $(d_2 + g_1) + (2g_2) + g_2 + 3C^2_3$. The $(2g_2)$ and the g_2 as in (53). Two C_3 have each a P'_2 at the P_4, a P'_1 at the discrete P_3 and a P'_1 on the $(2g_2)$. The other C_3 is gauche, has a P'_1 at the P_4, P'_1 at the $3P_3$ and meets the g_2 and the other $2C_3$ each again.

58. $d_3 + (2g_2) + 2g_2 + C^2_8$. The $(2g_2)$ passes through $2P_3$; each g_2, through the P_4. The C_8 has a P'_4 at the P_4 and a P'_3 at the discrete P_3. It meets each g_2 in a P'_1 and the $(2g_2)$ in $4P'_1$.

59. $d_3 + (2g_2) + 2g_2 + C^2_5 + C^2_3$. The $(2g_2)$ and the $2g_2$ as in (58). The C_5 has P'_2 at the P_4 and at the discrete P_3. It has a P'_1 on each g_2 and $3P'_1$ on the

$(2g_2)$. The C_3 has a P'_2 at the P_4 and a P'_1 at the discrete P_3. It meets the C_5 again and meets the $(2g_2)$ once.

60. $d_3 + (2g_2) + 2g_2 + 2C_3^2 + C_2^2$. The $(2g_2)$ and the $2g_2$ as in (58). Each C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and a P'_1 on the $(2g_2)$. The C_2 has P'_1 at the $3P_3$, meets each C_3 again and has a P'_1 on each g_2.

61. $d_3 + 2\ (2g_2) + C_6^2 + C_2^2$. One $(2g_2)$ has its pinch-points coincident at the P_4; the other has two pinch-points which are P_3 on the R_7. The C_6 has, at the P_4, a P'_3 with one branch touching the $(2g_2)$. It has $3P'_1$ on the other $(2g_2)$ and a P'_2 at the discrete P_3. The C_2 touches one $(2g_2)$ at the P_4, meets the other $(2g_2)$ in a P'_1 and has a P'_1 at the discrete P_3. It does not meet C_6 again.

62. $d_3 + 2\ (2g_2) + 2C_3^2 + C_2^2$. The $2(2g_2)$ as in (61). One C_3 touches one $(2g_2)$ at the P_4 and has P'_1 at the $3P_3$. The other C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3, meets the other C_3 again and meets the $(2g_2)$ through $2P_3$ once. The C_2 touches one $(2g_2)$ at the P_4, meets the other $(2g_2)$ once and has a P'_1 at the discrete P_3.

63. $d_3 + (3g_2) + (2g_2) + C_5^2 + C_2^2$. The $(3g_2)$ passes through the P_4; the $(2g_2)$ through $2P_3$. The C_5 has P'_2 at the P_4 and at the discrete P_3. It meets the $(3g_2)$ again and has $3P'_1$ on the $(2g_2)$. The C_2 touches the $(3g_2)$ at the P_4, has a P'_1 at the discrete P_3 and meets the $(2g_2)$ once.

64. $d_3 + (3g_2) + (2g_2) + C_3^2 + 2C_2^2$. The C_3 has a P'_2 at the P_4, a P'_1 at the discrete P_3 and a P'_1 on the $(2g_2)$. One C_2 touches the $(3g_2)$ at the P_4, has a P'_1 at the discrete P_3 and a P'_1 on the $(2g_2)$. The other C_2 has P'_1 at the $3P_3$, meets the C_3 again and meets the $(3g_2)$ once.

86. When the R_7 has g_3, we may take the g_3 for $y = z = 0$ in transformation II and transform the R_7 into a C_4 of the second kind having a P'_1 on $\bar{y} = \bar{z} = 0$. The R_7 has only $2P_3$ since only two trisecants of the C_4 cut $\bar{x} = \bar{y} = 0$. The scroll of bisecants is, as we have seen,* an R_9 which may break up into an R_6 and an R_3 or into $3R_3$. The three generators of the scroll of bisecants through the P'_1 of the C_4 on $\bar{y} = \bar{z} = 0$ transform into the triple point of the double curve at the P_4; the other three generators in $\bar{y} = 0$ transform into three other points on the g_3.

87. When the R_7 has a g_2 in addition to the g_3, the C_4 has a P'_2. The scroll of bisecants in an R_8 which may break up into an R_6 and a K_2, or into an R_5 and

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* See paragraph 75.

an R_3 or into $2R_3$ and a K_2. When the P_2' lies in $\bar{y}=0$ the g_2 becomes consecutive to the g_3.

88. When the tangents to the P_3' cut $\bar{x}=\bar{y}=0$, the R_7 has a $(2g_2)$. The scroll of bisecants is an R_7 which may break up into an R_5 and a K_2 or into an R_3 and $2K_2$.

65. $d_3+g_3+C_9^2$. The g_3 goes through the P_4. The C_9 has P_3' at the P_4 and at the $2P_3'$. It has $3P_1'$ on the g_3. $p'=1, 0$.

66. $d_3+g_3+C_6^2+C_3^2$. The g_3 goes through the P_4. The C_6 has P_2' at the P_4 and at the $2P_3$. It has $2P_1'$ on the g_3. The C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. It meets the C_6 twice and the g_3 once again.

67. $d_3+g_3+3C_3^2$. The g_3 goes through the P_4. Each C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. Each meets the g_3 and each of the other C_3 again.

68. $d_3+g_3+g_2+C_8^2$. The g_3 goes through the P_4; the g_2, through $2P_3$. The g_3 and g_2 may be consecutive. The C_8 has a P_3' at the P_4 and $3P_1'$ on the g_3. It has $2P_2'$ and a P_1' on the g_2. $p'=1, 0$.

69. $d_3+g_3+g_2+C_6^2+C_2^2$. The g_3 and the g_2 as in (68). The C_6 has a P_2' at the P_4 and at the $2P_3$. It meets the g_3 and the C_2 each in two more P_1'. The C_2 has a P_1' at the P_4, meets the g_3 again and has a P_1' on the g_2.

70. $d_3+g_3+g_2+C_5^2+C_3^2$. The g_3 and the g_2 as in (68). The C_5 has a P_2' at the P_4, $2P_1'$ on the g_3 and $3P_1'$ on the g_2. The C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. It meets the C_5 twice, and the g_3 once, again.

71. $d_3+g_3+g_2+2C_3^2+C_2^2$. The g_3 and the g_2 as in (68). The C_2 has a P_1' at the P_4, meets the g_3 again and has a P_1' on the g_2. Each C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. Each meets the g_3, the C_2 and the other C_3 once again.

72. $d_3+g_3+(2g_2)+C_7^2$. The g_3 goes through the P_4; the $(2g_2)$ through $2P_3$. The C_7 has a P_3' at the P_4, $3P_1'$ on the g_3 and $4P_1'$ on the $(2g_2)$. $p'=0$.

73. $d_3+g_3+(2g_2)+C_5^2+C_2^2$. The g_3 and the $(2g_2)$ as in (72). The C_5 has a P_2' at the P_4 and meets the g_3 twice, and the C_2 once, again. It has $3P_1'$ on the $(2g_2)$. The C_2 has a P_1' at the P_4 and meets the g_3 again. It has a P_1' on the $(2g_2)$.

74. $d_3+g_3+(2g_2)+C_3^2+2C_2^2$. The g_3 and the $(2g_2)$ as in (72). Each C_2 goes through the P_4, meets the g_3 again and has a P_1' on the $(2g_2)$. The C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. It meets the g_3 and each C_2 once again.

Seven Threefold Points.

89. The R_7 is transformed by I into a C_5 meeting $\bar{x} = \bar{y} = 0$ and passing through each fundamental point. The scroll of bisecants is an R_{12}. Taking temporarily a trisecant of the C_5 for $\bar{y} = \bar{z} = 0$ in transformation II, the C_5 is transformed into an R_6 with a d_2. From the known properties of the nodal curve of such an R_6,* it is seen that the R_{12} of bisecants may break up into an R_8 and an R_4 or into $3R_4$. $\bar{z} = \bar{w} = 0$ meets the C_5 in 0, 1, 2 points besides the fundamental points.

1. $d_3 + C_{12}^2$; $(d_2 + g_1) + C_{12}^2$; $(d_1 + 2g_1) + C_{12}^2$. The C_{12} has P_3' at the $7P_3$. $p' = 1, 0$.

2. $d_3 + C_8^2 + C_4^2$; $(d_2 + g_1) + C_8^2 + C_4^2$; $(d_1 + 2g_1) + C_8^2 + C_4^2$. The C_8 has P_2' at the $7P_3$. The C_4 has P_1' at the $7P_3$ and meets the C_8 again in $2P_1'$.

3. $d_3 + 3C_4^2$; $(d_2 + g_1) + 3C_4^2$; $(d_1 + 2g_1) + 3C_4^2$. Each C_4 has P_1' at the $7P_3$ and meets each of the other C_4 again in a P_1'.

90. When the R_7 has a g_2 it transforms into a C_5 with a P_2'. The scroll of bisecants is an R_{11}. In the same way as before it may be seen that the R_{11} may break up into an R_8 and an R_3 or into an R_7 and an R_4 or into $2R_4$ and an R_3. $\bar{z} = \bar{w} = 0$ may meet the C_5, in addition to the fundamental points, in a P_1' or at the P_2'.

91. When the tangents at the P_2' cut $\bar{x} = \bar{y} = 0$, the multiple generator is double torsal. The scroll of bisecants is an R_{10} which may break up into an R_7 and an R_3 or into an R_4 and $2R_3$.

4. $d_3 + g_2 + C_{11}^2$; $(d_2 + g_1) + g_2 + C_{11}^2$; $(d_1 + g_2) + C_{11}^2$. The C_{11} has P_3' at the five discrete P_3 and $2P_2'$ and a P_1' on the g_2. $p' = 1, 0$.

5. $d_3 + g_2 + C_8^2 + C_3^2$; $(d_2 + g_1) + g_2 + C_8^2 + C_3^2$; $(d_1 + g_2) + C_8^2 + C_3^2$. The C_8 has P_2' at the $7P_3$. The C_3 has P_1' at the five discrete P_3 and meets the C_8 again in $2P_1'$. It has a P_1' on the g_2.

6. $d_3 + g_2 + C_7^2 + C_4^2$; $(d_2 + g_1) + C_7^2 + C_4^2$; $(d_1 + g_2) + C_7^2 + C_4^2$. The C_7 has P_2' at the five discrete P_3 and $3P_1'$ on the g_2. The C_4 has P_1' at the $7P_3$ and meets the C_7 again in $2P_1'$.

7. $d_3 + g_2 + 2C_4^2 + C_3^2$; $(d_2 + g_1) + 2C_4^2 + C_3^2$; $(d_1 + g_2) + 2C_4^2 + C_3^2$. The C_3 has P_1' at the five discrete P_3 and a P_1' on the g_2. Each C_4 has P_1' at the $7P_3$. Each C_4 meets the other C_4 and the C_3 each again in a P_1'.

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*See Wiman, loc. cit. p. 31. Snyder, Amer. Journal of Math., Vol. XXVII, p. 80.

8. $d_3 + (2g_2) + C_{10}^2$; $(d_2 + g_1) + C_{10}^2$; $[d_1 + (2g_2)] + C_{10}^2$. The C_{10} has P_3' at the five discrete P_3 and $4P_1'$ on the $(2g_2)$. $p' = 0$.

9. $d_3 + (2g_2) + C_7^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + C_7^2 + C_3^2$; $[d_1 + (2g_2)] + C_7^2 + C_3^2$. The C_7 has P_2' at the five discrete P_3 and $3P_1'$ on the $(2g_2)$. The C_3 has P_1' at the five discrete P_3 and meets the C_7 again. It has a P_1' on the $(2g_2)$.

10. $d_3 + (2g_2) + C_4^2 + 2C_3^2$; $(d_2 + g_1) + C_4^2 + 2C_3^2$; $(d_1 + g_2) + C_4^2 + 2C_3^2$. The C_4 has P_1' at the $7P_3$. Each C_3 has P_1' at the five discrete P_3, a P_1 on the $(2g_2)$ and meets the C_4 again in a P_1'.

92. When the R_7 has $2g_2$, the C_5 has $2P_2'$. The scroll of bisecants is an R_{10} which, it is readily seen, may break up into an R_8 and a K_2, an R_7 and an R_3, an R_6 and an R_4, $2R_4$ and a K_2 or into an R_4 and $2R_3$. $\bar{z} = \bar{w} = 0$ may be a trisecant of the C_5 except when the scroll of bisecants has a component K_2.

93. When the tangents at one P_2' cut $\bar{x} = \bar{y} = 0$, one multiple generator of the R_7 is a $(2g_2)$. The scroll of bisecants of the C_5 in an R_9 which may break up into an R_7 and a K_2, an R_6 and an R_3, an R_4 and an R_3 and an R_2 or into $3R_3$. $\bar{z} = \bar{w} = 0$ be a trisecant except when the scroll of bisecants has a component K_2.

94. When the tangents of both P_2' cut $\bar{x} = \bar{y} = 0$, then both multiple generators are double torsal. The scroll of bisecants is an R_6 and a K_2 or $2R_3$ and a K_2.

11. $d_3 + 2g_2 + C_{10}^2$; $(d_2 + g_1) + 2g_2 + C_{10}^2$. The C_{10} has P_3' at the three discrete P_3 and $2P_2'$ and a P_1' on each g_2. $p' = 1, 0$.

12. $d_3 + 2g_2 + C_8^2 + C_2^2$. The C_8 has P_2' at the $7P_3$. The C_2 has P_1' at the three discrete P_3, meets the C_8 again in $2P_1'$ and meets each g_2 once.

13. $d_3 + 2g_2 + C_7^2 + C_3^2$; $(d_2 + g_1) + 2g_2 + C_7^2 + C_3^2$. The C_7 has P_2' at the three discrete P_3 and at the $2P_3$ on one g_2. It has $3P_1'$ on the other g_2. The C_3 has P_1' at the three discrete P_3 and at the $2P_3$ on the latter g_2. It meets the C_7 again in $2P_1'$ and meets the former g_2 once.

14. $d_3 + 2g_2 + C_6^2 + C_4^2$; $(d_2 + g_1) + 2g_2 + C_6^2 + C_4^2$. The C_6 has P_2' at the three discrete P_3 and $3P_1'$ on each g_2. The C_4 has P_1' at the $7P_3$ and meets the C_6 again in $2P_1'$.

15. $d_3 + 2g_2 + 2C_4^2 + C_2^2$. The C_2 has P_1' at the three discrete P_3 and meets each g_2 once. Each C_4 has P_1' at the $7P_3$ and meets the other C_4 and the C_2 each again.

16. $d_3 + 2g_2 + C_4^2 + 2C_3^2$; $(d_2 + g_1) + 2g_2 + 2C_3^2$. The C_4 has P_1' at the $7P_3$. Each C_3 has P_1' at the three discrete P_3 and at the $2P_3$ on one g_2 and meets the

other g_2 in a P_1'. Each two of the curves C_4, C_3 and C_3 have another P_1' common.

17. $d_3 + (2g_2) + g_2 + C_9^2$; $(d_2 + g_1) + (2g_2) + g_2 + C_9^2$. The C_9 has P_3' at the three discrete P_3, $2P_2'$ and a P_1' on the g_2 and $4P_1'$ on the $(2g_2)$. $p' = 0$.

18. $d_3 + (2g_2) + g_2 + C_7^2 + C_2^2$. The C_7 has P_2' at the three discrete P_3, $2P_2'$ on the g_2 and $3P_1'$ on the $(2g_2)$. The C_2 has P_1' at the three discrete P_3, meets the C_7 in another P_1', has a P_1' on the g_2 and one on the $(2g_2)$.

19. $d_3 + (2g_2) + g_2 + C_6^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + g_2 + C_6^2 + C_3^2$. The C_6 has P_2' at the three discrete P_3, $3P_1'$ on the g_2 and $3P_1'$ on the $(2g_2)$. The C_3 goes through the three discrete P_3 and the $2P_3$ on the g_2. It has another P_1' on the C_6 and a P_1' on the $(2g_2)$.

20. $d_3 + (2g_2) + g_2 + C_4^2 + C_3^2 + C_2^2$. The C_4 has P_1' at the $7P_3$ and meets the C_3 and the C_2 each again. The C_3 has P_1' at the three discrete P_3 and at the $2P_3$ on the g_2. It has a P_1' on the $(2g_2)$. The C_2 has P_1' at the three discrete P_3 and P_1' on the g_2 and on the $(2g_2)$.

21. $d_3 + (2g_2) + g_2 + 3C_3^2$; $(d_2 + g_1) + 2g_2 + g_2 + 3C_3^2$. Each C_3 has P_1' at the three discrete P_3. One C_3 has $2P_1'$ on the $(2g_2)$, a P_1' on the g_2 and another P_1' on each of the other C_3. Each of the other C_3 has $2P_1'$ on the g_2 and one P_1' on the $2g_2$.

22. $d_3 + 2(2g_2) + C_6^2 + C_2^2$. The C_6 has P_2' at the three discrete P_3 and $3P_1'$ on each $(2g_2)$. The C_2 has P_1' at the three discrete P_3 and has a P_1' on each $(2g_2)$.

23. $d_3 + 2(2g_2) + 2C_3^2 + C_2^2$. The $2C_3$ and C_2 each have P_1' at the three discrete P_3. Each C_3 has $2P_1'$ on one $(2g_2)$, and one P_1' on the other $(2g_2)$ and meets the other C_3 again. The C_2 has a P_1' on each $(2g_2)$.

95. When the R_7 has $3g_2$, take one g_2 for $y = z = 0$ in transformation II. The R_7 is transformed into a C_5 with $2P_2'$. The C_5 meets $\bar{x} = \bar{y} = 0$ once and $\bar{y} = \bar{z} = 0$ twice. We have seen that the scroll of bisecants to such a C_5 is an R_{10} which may break up into an R_8 and a K_2, an R_7 and an R_3, an R_6 and an R_4, $2R_4$ and a K_2 or into an R_4 and $2R_3$. $\bar{y} = \bar{z} = 0$ is a simple generator of the scroll of bisecants. When the C_5 passes through $\bar{x} = \bar{y} = \bar{z} = 0$, the R_7 has a g_1 coinciding with the directrix. One, two, or all three multiple generators may be double torsals. In the latter case the tangents at both double points meet $\bar{x} = \bar{y} = 0$ and the C_5 touches $\bar{y} = 0$ twice on $\bar{y} = \bar{z} = 0$.

96. Whenever the scroll of bisecants has a component K_2 of which $\bar{y} = \bar{z} = 0$ is a generator, the R_7 has a double contact directrix $(\delta_{2,1} + g_1)$. To the two points on an arbitrary generator of the K_2 corresponds two generators intersecting on $\bar{x} = \bar{y} = 0$ and coplanar with $\bar{x} = \bar{y} = 0$. The directrix counts once as a generator.

24. $d_3 + 3g_2 + C_9^2$; $(d_2 + g_1) + 3g_2 + C_9^2$. The C_9 has a P_3' at the discrete P_3 and $2P_2'$ and a P_1' on each g_2. $p' = 1, 0$.

25. $d_3 + 3g_2 + C_7^2 + C_2^2$. The C_7 has P_2' at the discrete P_3 and at both P_3 on $2g_2$. It meets the other g_2 in $3P_1'$. The C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the latter g_2. It meets each of the other g_2 once and meets the C_7 in two other P_1'.

26. $d_3 + 3g_2 + C_6^2 + C_3^2$; $(d_2 + g_1) + 3g_2 + C_6^2 + C_3^2$. The C_6 has P_2' at the discrete P_3 and at the $2P_3$ on one g_2 and $3P_1'$ on each of the other g_2. The C_3 has P_1' at the $2P_3$ on each of the latter $2g_2$, a P_1' on the former g_2, a P_1' at the discrete P_3 and meets the C_6 again in $2P_1'$.

27. $d_3 + 3g_2 + C_5^2 + C_4^2$; $(d_2 + g_1) + 3g_2 + C_5^2 + C_4^2$. The C_5 has a P_2' at at the discrete P_3 and $3P_1'$ on each g_2. The C_4 has P_1' at the $7P_3$ and two other P_1' on the C_5.

28. $d_3 + 3g_2 + C_4^2 + C_3^2 + C_2^2$. The C_4 has P_1' on the $7P_3$. The C_3 has P_1' at the discrete P_3 and at both P_3 on $2g_2$ and meets the other g_2 once. The C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the latter g_2. It meets each of the other g_2 once. Each two of the curves C_4, C_3, and C_2 have another point in common.

29. $d_3 + 3g_2 + 3C_3^2$; $(d_2 + g_1) + 3g_2 + 3C_3^2$. Each C_3 has P_1' at the discrete P_3 and at both P_3 on $2g_2$, meets the other g_2 once again and has another P_1' on each of the other C_3.

30. $(\delta_{2,1} + g_1) + 3g_2 + C_8^2$. The C_8 has P_2' at the $7P_3$.

31. $(\delta_{2,1} + g_1) + 3g_2 + 2C_4^2$. Each C_4 has P_1' at the $7P_3$ and another P_1' on the other C_4.

32. $d_3 + (2g_2) + 2g_2 + C_8^2$; $(d_2 + g_1) + (2g_2) + 2g_2 + C_8^2$. The C_8 has a P_3' at the discrete P_3, $2P_2'$ and a P_1' on each g_2 and $4P_1'$ on the $(2g_2)$. $p' = 0$.

33. $d_3 + (2g_2) + 2g_2 + C_6^2 + C_2^2$. The C_6 has P_2' at the discrete P_3, and at the $2P_3$ on one g_2. It has $3P_1'$ on the other g_2 and on the $(2g_2)$. The C_2 has a P_1' at the discrete P_3, $2P_1'$ on the second g_2, a P_1' on the former g_2 and on the $(2g_2)$ and another P_1' on the C_6.

34. $d_3 + (2g_2) + 2g_2 + C_5^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + 2g_2 + C_5^2 + C_3^2$. The C_5 has a P_2' at the discrete P_3, P_1' at the other $6P_3$ and meets the C_3, the $2g_2$ and the $(2g_2)$ each again. The C_3 has P_1' at the $7P_3$.

35. $d_3 + (2g_2) + 2g_2 + C_4^2 + 2C_2^2$. The C_4 has P_1' at the $7P_3$. Each C_2 has P_1' at the discrete P_3 and at the $2P_3$ on one g_2, meets the other g_2 and the $(2g_2)$ each once and has another P_1' on the C_4.

36. $d_3 + (2g_2) + 2g_2 + 2C_3^2 + C_2^2$. One C_3 has P_1' at the discrete P_3, at the $2P_3$ on the $(2g_2)$ and at the $2P_3$ on one g_2. It has a P_1' on the other g_2 and meets the C_2 and the other C_3 each again. The other C_3 has P_1' at the discrete P_3 and at the $2P_3$ on each g_2 and has a P_1' on the $(2g_2)$. The C_2 has P_1' at the discrete P_3 and at the $2P_2'$ on the second g_2. It has a P_1' on the g_2 and one on the $(2g_2)$.

37. $d_3 + (2g_2) + 2g_2 + 2C_3^2 + C_2^2$. Each C_3 has P_1' at the discrete P_3 and at the $2P_3$ on each g_2. Each has a P_1' on the $(2g_2)$. The C_2 has a P_1' at the discrete P_3 and meets each C_3 again. It meets the $(2g_2)$ twice and each g_2 once.

38. $(\delta_{2,1} + g_1) + (2g_2) + 2g_2 + C_7^2$. The C_7 has P_2' at $5P_3$, P_1' at the $2P_3$ on the $(2g_2)$. It meets the $(2g_2)$ once again.

39. $(\delta_{2,1} + g_1) + (2g_2) + 2g_2 + C_4^2 + C_3^2$. The C_4 has P_1' at the $7P_3$. The C_3 has P_1' at $5P_3$, meets the C_4 again and has a P_1' on the $(2g_2)$.

40. $d_3 + 2(2g_2) + g_2 + C_5^2 + C_2^2$. The C_5 has a P_2' at the discrete P_3 and P_1' at the other $6P_3$. It meets the g_2 and each $(2g_2)$ again. The C_2 has P_1' at the discrete C_3 and at the $2P_3$ on the g_2. It has a P_1' on each $(2g_2)$.

41. $d_3 + 2(2g_2) + g_2 + C_3^2 + 2C_2^2$. The C_3 has P_1' at the discrete P_3, at the $2P_3$ on the g_2 and at those on one $(2g_2)$. It has a P_1' on the other $(2g_2)$. One C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the second $(2g_2)$. It meets the C_3 again, has a P_1' on the $(2g_2)$ and one on the g_2. The other C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the g_2. It has a P_1' on each $(2g_2)$.

42. $(\delta_{2,1} + g_1) + 2(2g_2) + g_2 + C_6^2$. The C_6 has P_2' at the P_3 on the directrix and at the $2P_3$ on the g_2. It has $3P_1'$ on each $(2g_2)$.

43. $(\delta_{2,1} + g_1) + 2(2g_2) + g_2 + 2C_3^2$. Each C_3 has P_1' at the P_3 on the directrix and at the $2P_3$ on the g_2. Each has $2P_1'$ on one $(2g_2)$ and $1P_1'$ on the other.

44. $d_3 + 3(2g_2) + 3C_2^2$. Each C_2 goes through the discrete P_3 and the $2P_3$ on one $(2g_2)$. Each has a P_1' on each of the other $(2g_2)$.

$p=1$.

Fourfold Point.

97. The R_7 is transformed by I into a C_4 of the first kind passing through one fundamental point. The scroll of bisecants is an R_8 of genus 3, 2, 1. It may break up into an R_6 and a K_2 or into an R_4 and $2K_2$.

98. When the fundamental point not on the C_4' lies on the scroll of bisecants, the R_7 has a g_2 through the P_4. When it lies at the intersection of two generators of the scroll of bisecants, the R_7 has $2g_2$ through the P_4. When the fundamental point lies on a $(2g_2)$ of the scroll of bisecants the R_7 has a $(2g_2)$, or, in particular, if it lies at a pinch-point, a $(3g_2)$, through the P_4.

The R_7 cannot have a g_2 not passing through the P_4.

1. $d_3+C_{11}^2$; $(d_2+g_1)+C_{11}^2$. The C_{11} has a P_6' at the P_4 and a P_3' at the P_3. $p'=3, 2, 1$.

2. $d_3+C_8^2+C_8^2$; $(d_2+g_1)+C_8^2+C_8^2$. The C_8 has a P_4' at the P_4 and a P_3' at the P_3. p' for the C_8 is 2 or 1. The C_8 has a P_2' at the P_4, a P_1' at the P_3 and two other P_1' on the C_8.

3. $d_3+C_5^2+2C_8^2$; $(d_2+g_1)+C_5^2+2C_8^2$. Each curve has a P_2' at the P_4 and a P_1' at the P_3. The C_5 meets each C_8 again in $2P_1'$. p' for the C_5 is 1.

4. $d_3+g_2+C_{10}^2$; $(d_2+g_1)+g_2+C_{10}^2$. The C_{10} has a P_5' at the P_4 and a P_3' at the P_3. It meets the g_2 again in a P_1'. $p'=3, 2$, or 1.

5. $d_3+g_2+C_7^2+C_8^2$; $(d_2+g_1)+g_2+C_7^2+C_8^2$. The C_7 has a P_3' at the P_4 and a P_2' at the P_3. It meets the g_2 again. $p'=2, 1$. The C_8 has a P_2' at the P_4, a P_1' at the P_3, and two other P_1' on the C_7.

6. $d_3+g_2+C_8^2+C_2^2$. The C_8 has a P_4' at the P_4 and a P_2' at the P_3. $p'=2$ or 1. The C_2 has P_1' at the P_4 and at the P_3. It meets the C_8 twice and the g_2 once again.

7. $d_3+g_2+C_4^2+2C_8^2$; $(d_2+g_1)+g_2+C_4^2+2C_8^2$. The C_4 has P_1' at the P_4 and at the P_3. It meets the g_2 again. $p'=1$. Each C_8 has a P_2' at the P_4, a P_1' at the P_3 and meets the C_4 twice again.

8. $d_3+g_2+C_5^2+C_8^2+C_2^2$. The C_5 and C_8 each have a P_2' at the P_4 and a P_1' at the P_3. The C_2 has P_1' at the P_4' and at the P_3 and meets the g_2 again. The C_5 meets the C_8 and C_2 each twice again. Its genus is 1.

9. $d_3+2g_2+C_9^2$; $(d_2+g_1)+2g_2+C_9^2$. The C_9 has a P_4' at the P_4, and a P_3' at the P_3. It meets each g_2 again. $p'=3, 2, 1$.

10. $d_3 + 2g_2 + C_6^2 + C_3^2$; $(d_2 + g_1) + 2g_2 + C_6^2 + C_3^2$. The C_6 has P_2' at the P_4 and P_3, and meets each g_2 again. $p' = 2$ or 1. The C_3 has a P_2' at the P_4, a P_1' at the P_3, and meets the C_6 twice again.

11. $d_3 + 2g_2 + 3C_3^2$; $(d_2 + g_1) + 2g_2 + 3C_3^2$. One C_3 has a P_1' at the P_3 and one on each g_2. Its genus is 1. Each of the other two C_3 has a P_2' at the P_4, a P_1' at the P_3 and two other P_1' on the first C_3.

12. $d_3 + (2g_2) + C_9^2$. The C_9 has at the P_4 a P_4' with two branches touching the $(2g_2)$. It has a P_3' at the P_3. $p' = 2$ or 1.

13. $d_3 + (2g_2) + C_6^2 + C_3^2$. The C_6 has two consecutive P_2' at the P_4 and a P_2' at the P_3. $p' = 1$. The C_3 has a P_2' at the P_4, and a P_1' at the P_3. It meets the C_6 again in $2P_1'$.

14. $d_3 + (2g_2) + C_7^2 + C_2^2$. The C_7 has at the P_4 a P_3' with one branch touching the $(2g_2)$ and a P_2' at the P_3. $p' = 2$ or 1. The C_2 touches the $(2g_2)$ at the P_4, has a P_1' at the P_3 and meets the C_7 once again.

15. $d_3 + (2g_2) + C_4^2 + C_3^2 + C_2^2$. The C_4 touches the $(2g_2)$ at the P_4 and has a P_1' at the P_3. $p' = 1$. The C_3 has a P_2' at the P_4, a P_1' at the P_3, and meets the C_4 twice again. The C_2 touches the $(2g_2)$ at the P_4, has a P_1' at the P_3, and meets the C_4 once again.

16. $d_3 + (3g_2) + C_8^2$; $(d_2 + g_1) + (3g_2) + C_8^2$. The C_8 has, at the P_4, a P_3' with one branch touching the $(3g_2)$. It has a P_3' at the P_3 and meets the $(3g_2)$ again. $p' = 2$, or 1.

17. $d_3 + (3g_2) + C_5^2 + C_3^2$; $(d_2 + g_1) + (3g_2) + C_5^2 + C_3^2$. The C_5 touches the $(3g_2)$ at the P_4, meets the $(3g_3)$ again, and has a P_2' at the P_3. $p' = 1$. The C_3 has a P_2' at the P_4, a P_1' at the P_3 and meets the C_5 twice again.

18. $d_3 + (3g_2) + C_6^2 + C_2^2$. The C_6 has P_2' at the P_4 and at the P_3 and meets the $(3g_2)$ again. $p' = 2$ or 1. The C_2 touches the $(3g_2)$ at the P_4, has a P_1' at the P_3, and meets the C_6 once again.

19. $d_3 + (3g_2) + 2C_3^2 + C_2^2$. One C_3 has a P_1' at the P_3 and a P_1' on the $(3g_2)$. $p' = 1$. The other C_3 has a P_2' at the P_4, a P_1' at the P_3, and meets the other C_3 in $2P_1'$. The C_2 touches the $(3g_2)$ at the P_4, has a P_1' at the P_3, and meets the first C_3 again.

99. When the R_7 has a double contact directrix, the P_3 is on the directrix. Taking the P_4 and a P_2 for fundamental points the R_7 is transformed into a C_5 with a P_3' at one fundamental point and lying on an R_3 which has $\bar{x} = \bar{y} = 0$ for d_1 and $\bar{z} = \bar{w} = 0$ for d_2. The residual scroll of bisecants is an R_7 of genus 2, or 1,

or an R_4 of genus 1 and an R_3 or an R_5 of genus 1 and a K_2. The fundamental point not on the C_5 may be at a pinch-point of a $(2g_2)$ of the scroll of bisecants, giving rise to a $(3g_2)$.

20. $(\delta_{2,1}+g_1)+2g_2+C_8^2$. The C_8 has a P_4' at the P_4 and a P_2' at the P_3. $p'=2$ or 1.

21. $(\delta_{2,1}+g_1)+2g_2+C_5^2+C_3^2$. The C_5 and C_3 each have a P_2' at the P_4 and a P_1' at the P_3. They meet again in $2P_1'$. p' for the C_5 is 1.

22. $(\delta_{2,1}+g_1)+(3g_2)+C_7^2$. The C_7 has, at the P_4, a P_3' with one branch touching the $(3g_2)$. It has a P_2' at the P_3. $p'=2$ or 1.

23. $(\delta_{2,1}+g_1)+(3g_2)+C_4^2+C_3^2$. The C_4 touches the $(3g_2)$ at the P_4 and has a P_1' at the P_3. $p'=1$. The C_3 has a P_2' at the P_4, a P_1' at the P_3 and meets the C_4 twice again.

100. When the R_7 has a g_3 it may be transformed by II into a C_4 of the first kind cutting $\bar{y}=\bar{z}=0$ once. We have seen that the scroll of the bisecants is an R_8 which may break up into an R_6 and a K_2 or into an R_4 and $2K_2$.

24. $d_3+g_3+C_8^2$. The C_8 has a P_3' at the P_4 and $3P_1'$ on the g_3. $p'=3$, 2, or 1.

25. $d_3+g_3+C_6^2+C_2^2$. The C_6 has a P_2' at the P_4 and $2P_1'$ on the g_3. $p'=2$ or 1. The C_2 has a P_1' at the P_4 and meets the C_6 twice, and the g_3 once again.

26. $d_3+g_3+C_4^2+2C_2^2$. Each curve has a P_1' at the P_4 and meets the g_3 again. The C_4 meets each C_2 twice again. p' for the C_4 is 1.

Five Threefold Points.

101. The R_7 is transformed by I into a C_5 of genus 1 meeting $\bar{x}=\bar{y}=0$ and passing through both fundamental points. Taking, temporarily, a trisecant cutting $\bar{x}=\bar{y}=0$ for $\bar{y}=\bar{z}=0$ in transformation II, the C_5 may be transformed into an R_6 with a d_2. From known properties of the nodal curve of such of R_6,* it follows that the R_{11} of bisecants to the C_5 may break up into an R_8 and an R_3 or into an R_5 and $2R_3$. $\bar{z}=\bar{w}=0$ may meet the C_5 in one point not a fundamental point.

102. When the R_7 has a g_2 the C_5 has a P_2'. The scroll of bisecants is an R_{10} which may break up into an R_7 and an R_3, into an R_3 and a K_2, into an R_4 and $2R_3$ or into an R_5, an R_3 and a K_2.

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*See Wiman loc. cit. p. 52. Snyder, Amer. Journal of Math. Vol. 25, p. 96.

103. When the multiple generator is a $(2g_2)$ the tangents at the P'_2 cut $\bar{x}=\bar{y}=0$. The scroll of bisecants is an R_9 which may break up into an R_6 and an R_3, an R_7 and a K_2, or an R_4, an R_3, and a K_2.

104. When the R_7 has $2g_2$, take one of them for $y=z=0$ in transformation II. The R_7 goes into a C_5 with a P'_2 as obtained for the transform of an R_7 with one g_2. It meets $\bar{x}=\bar{y}=0$ once and $\bar{y}=\bar{z}=0$ twice. When the tangents at the P'_2 meet $\bar{x}=\bar{y}=0$ one multiple generator is double torsal. When in addition $\bar{y}=\bar{z}=0$ joins the points of tangency of two tangents lying in $\bar{y}=0$, both multiple generators are double torsal.

105. When the R_7 has two multiple generators, it may have a double contact directrix. This happens when the scroll of bisecants has a component K_2 on which $\bar{y}=\bar{z}=0$ is a generator.

1. $d_3+C^2_{11}$; $(d_2+g_1)+C^2_{11}$. The C_{11} has P'_3 at the $5P_3$. $p'=3$, 2, or 1.

2. $d_3+C^2_8+C^2_3$; $(d_2+g_1)+C^2_8+C^2_3$. The C_8 has P'_2 at the $5P_3$. $p'=2$ or 1. The C_3 has P'_1 at the $5P_3$ and meets the C_8 again in $2P'_1$.

3. $d_3+C^2_5+2C^2_3$; $(d_2+g_1)+C^2_5+2C^2_3$. All three curves have P'_1 at the $5P_3$. The C_5 meets each C_3 again in $2P'_1$ and is of genus 1.

4. $d_3+g_2+C^2_{10}$; $(d_2+g_1)+g_2+C^2_{10}$. The C_{10} has P'_3 at the three discrete P_3 and $2P'_2$ and a P'_1 on the g_2. $p'=3$, 2, or 1.

5. $d_3+g_2+C^2_8+C^2_2$. The C_8 has P'_2 at the $5P_3$. $p'=2$ or 1. The C_2 has P_1 at the three discrete P_3, meets the C_8 twice again and meets the g_2 once.

6. $d_3+g_2+C^2_7+C^2_3$; $(d_2+g_1)+g_2+C^2_7+C^2_3$. The C_7 has P'_2 at the three discrete P_3 and $3P'_1$ on the g_2. $p'=2$ or 1. The C_3 has P'_1 at the $5P_3$ and meets the C_7 twice again.

7. $d_3+g_2+C^2_4+2C^2_3$; $(d_2+g_1)+g_2+C^2_4+2C^2_3$. The C_4 has P'_1 at the three discrete P_3 and a P'_1 on the g_2. $p'=1$. Each C_3 has P'_1 at the $5P_3$ and meets the C_4 twice again.

8. $d_3+g_2+C^2_5+C^2_3+C^2_2$. The C_5 has P'_1 at the $5P_3$ and meets the C_3 and the C_2 each twice again. $p'=1$. The C_3 has P'_1 at the $5P_3$. The C_2 has P'_1 at the three discrete P_3 and has a P'_1 on the g_2.

9. $d_3+(2g_2)+C^2_9$; $(d_2+g_1)+(2g_2)+C^2_9$. The C_9 has P'_3 at the three discrete P_3 and $4P'_1$ on the $(2g_2)$. $p'=2$ or 1.

10. $d_3+(2g_2)+C^2_7+C^2_2$. The C_7 has P'_2 at the three discrete P_3 and

$3P_1'$ on the $(2g_2)$. $p'=2$ or 1. The C_2 has P_1' at the three discrete P_3, meets the C_7 once again and meets the $(2g_2)$ once.

11. $d_3+(2g_2)+C_6^2+C_3^2$: $(d_2+g_1)+(2g_2)+C_6^2+C_3^2$. The C_6 has P_1' at the three discrete P_3 and meets the $(2g_2)$ twice. $p'=1$. The C_3 has P_1' at the $5P_3$ and meets the C_6 twice again.

12. $d_3+(2g_2)+C_4^2+C_3^2+C_2^2$. Each curve has P_1' at the $3P_3$. The C_3 goes through the pinch-points of the $(2g_2)$. The C_4 and C_2 each meet it once. The C_4 meets the C_3 twice and the C_2 once again. p' for the C_4 is one.

13. $d_3+2g_2+C_9^2$; $(d_2+g_1)+2g_2+C_9^2$. The C_9 has a P_3' at the discrete P_3 and $2P_2'$ and a P_1' on each g_2. $p'=3$, 2 or 1.

14. $d_3+2g_2+C_7^2+C_2^2$. The C_7 has P_2' at the discrete P_3 and at the $2P_3$ on one g_2. It has $3P_1'$ on the other g_2. $p'=2$ or 1. The C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the second g_2. It meets the C_7 in two other P_1' and meets the first g_2 once.

15. $d_3+2g_2+C_6^2+C_3^2$. The C_6 has a P_2' at the discrete P_3 and $3P_1'$ on each g_2. $p'=2$ or 1. The C_3 has P_1' at the $5P_3$ and meets the C_6 twice again.

16. $d_3+2g_2+C_5^2+2C_2^2$. The C_5 has P_1' at the $5P_3$. $p'=1$. Each C_2 has a P_1' at the discrete P_3 and at the $2P_3$ on one g_2. Each meets the C_5 twice again and meets the other g_2 once.

17. $d_3+2g_2+C_4^2+C_3^2+C_2^2$. Each curve has a P_1' at the discrete P_3. The C_4 and C_2 each have $2P_1'$ on one g_2 and one P_1' on the other. The C_3 has $2P_1'$ on each g_2. The C_4 meets the C_3 and C_2 each twice again. p' for the C_4 is 1.

18. $d_3+2g_2+3C_3^2$; $(d_2+g_1)+2g_2+3C_3^2$. Each C_3 has a P_1' at the discrete P_3. One C_3 meets each g_2 once and is of genus 1. Each of the other C_3 is gauche, has $2P_1'$ on each g_2, and meets the first C_3 twice again.

19. $(\delta_{2,1}+g_1)+2g_2+C_8^2$. The C_8 has P_2' at the $5P_3$. $p'=2$ or 1.

20. $(\delta_{2,1}+g_1)+2g_2+C_5^2+C_3^2$. The C_5 and C_3 each have P_1' at the $5P_3$ and meet again in $2P_1'$. The C_5 is of genus 1. The C_3 is gauche.

21. $d_3+(2g_2)+g_2+C_8^2$; $(d_2+g_1)+(2g_2)+g_2+C_8^2$. The C_8 has a P_3' at the discrete P_3, $2P_2'$ and a P_1' on the g_2 and $4P_1'$ on the $(2g_2)$. $p'=2$ or 1.

22. $d_3+(2g_2)+g_2+C_6^2+C_2^2$. The C_6 has a P_2' at the discrete P_3 and $3P_1'$ each on the g_2 and the $(2g_2)$. $p'=2$ or 1. The C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the g_2, meets the C_6 once again and meets the $(2g_2)$ once.

23. $d_3+(2g_2)+g_2+C_6^2+C_2^2$. The C_6 has P_2' at the discrete P_3 and at

the $2P_3$ on the g_2. It meets the $(2g_2)$ twice. $p' = 1$. The C_2 has P_1' at the discrete P_3 and the $2P_3$ on the $(2g_2)$. It meets the C_6 twice again.

24. $d_3 + (2g_2) + g_2 + C_5^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + g_2 + C_5^2 + C_3^3$. The C_5 has a P_1' at the discrete P_3, $3P_1'$ on the g_2 and $2P_1'$ on the $(2g_2)$. $p' = 1$. The C_3 is gauche. It has P_1' at the $5P_3$ and meets the C_5 twice again.

25. $d_3 + (2g_2) + g_2 + C_4^2 + 2C_2^2$. Each curve has a P_1' at the discrete P_3. One C_2 has $2P_1'$ on the g_2 and one P_1' on the $(2g_2)$. The other has a P_1' on the g_2 and $2P_1'$ on the $(2g_2)$. The C_4 has $2P_1'$ on the g_2, one P_1' on the $(2g_2)$, and meets each C_2 twice again. Its genus is 1.

26. $d_3 + (2g_2) + g_2 + 2C_3^2 + C_2^2$. Each curve has a P_1' at the discrete P_3. The C_2 has $2P_1'$ on the g_2 and one P_1' on the $(2g_2)$. One C_3 is gauche. It has $2P_1'$ each on the g_2 and the $(2g_2)$. The other C_3 is of genus 1. It meets the g_2 and the $(2g_2)$ each once, meets the C_3 twice and the C_2 once again.

27. $(\delta_{2,1} + g_1) + (2g_2) + g_2 + C_7^2$. The C_7 has P_2' at the P_3' on the directrix and the $2P_3'$ on the g_2. It has $3P_1'$ on the $(2g_2)$. $p' = 2$ or 1.

28. $(\delta_{2,1} + g_1) + (2g_2) + g_2 + C_4^2 + C_3^2$. Each curve has P_1' at the P_3 on the directrix and the $2P_3$ on the g_2. Each meets the $(2g_2)$ twice. They meet again in $2P_1'$. p' for the C_4 is 1. The C_3 is gauche.

29. $d_3 + 2(2g_2) + C_7^2$. The C_7 has a P_3' at the discrete P_3 and $4P_1'$ on each $(2g_2)$. $p' = 1$.

30. $d_3 + 2(2g_2) + C_5^2 + C_2^2$. The C_5 has a P_2' at the discrete P_3. It has $2P_1'$ on one $(2g_2)$ and $3P_1'$ on the other. $p' = 1$. The C_2 has $2P_1'$ on one $(2g_2)$ and one P_1' on the other. It meets the C_5 once again.

31. $d_3 + 2(2g_2) + C_3^3 + 2C_2^2$. Each curve has a P_1' at the discrete P_3. The C_3 has a P_1' on each $(2g_2)$. $p' = 1$. Each C_2 has $2P_1'$ on one $(2g_2)$ and one P_1' on the other $(2g_2)$. Each meets the C_3 once again.

32. $(\delta_{2,1} + g_1) + 2(2g_2) + C_6^2$. The C_6 has a P_2' at the P_3 on the directrix and $3P_1'$ on each $(2g_2)$. $p' = 2$ or 1.

33. $(\delta_{2,1} + g_1) + 2(2g_2) + 2C_3^2$. Each C_3 has a P_1' at the P_3 on the directrix. One is of genus 1 and meets each $(2g_2)$ once. The other C_3 is gauche, goes through the pinch points of each $(2g_2)$ and meets the plane C_3 twice again.

$$p = 2$$

106. The R_7 may be transformed by I into a C_5 of genus two, which meets $\bar{x} = \bar{y} = 0$ once and passes simply through each fundamental point. The C_5 has

a trisecant meeting $\bar{x} = \bar{y} = 0$ and may therefore be transformed by II into an R_8.* Hence we find that the R_{10} of bisecants to the C_5 is of genus 5, 4, 3 or 2 and may break up into an R_8 of genus 4, 3 or 2, and a K_2 into an R_6 and an R_4 each of genus 1 or into $2R_4$ of genus 1 and a K_2.

107. When the R_7 has a g_2, it transforms by II into a C_5 of the same kind as above. When the multiple generator is a $(2g_2)$ then $\bar{y} = \bar{z} = 0$ joins the points of tangency of two tangents lying in $\bar{y} = 0$.

1. $d_8 + C^2_{10}$; $(d_2 + g_1) + C^2_{10}$. The C_{10} has P'_8 at the $3P_8$. $p' = 5, 4, 3$ or 2.

2. $d_8 + C^2_8 + C^2_2$. The C_8 has P'_2 at the $3P_8$. $p' = 4$, 3 or 2. The C_2 has P'_1 at the $3P_8$ and meets the C_8 twice again.

3. $d_8 + C^2_6 + C^2_4$; $(d_2 + g_1) + C^2_6 + C^2_4$. The C_6 has P'_2 and the C_4, P'_1 at the $3P_8$. They meet again in $4P'_1$. Each is of genus 1.

4. $d_8 + 2C^2_4 + C^2_2$. Each C_4 meets the C_2 once and the other C_4 thrice again. Each C_4 is of genus 1.

5. $d_8 + g_2 + C^2_9$; $(d_2 + g_1) + g_2 + C^2_9$. The C_9 has a P'_8 at the discrete P_8 and $2P'_2$ and a P'_1 on the g_2. $p' = 5$, 4, 3 or 2.

6. $d_8 + g_2 + C^2_7 + C^2_2$. The C_7 has a P'_2 at the discrete P_8 and $3P'_1$ on the g_2. $p' = 4$, 3, or 2. The C_2 has a P'_1 at the $3P_3$ and meets the C_7 again in $2P'_1$.

7. $d_8 + g_2 + C^2_6 + C^2_3$; $(d_2 + g_1) + g_2 + C^2_6 + C^2_3$. The C_6 has P'_2 at the $3P_8$. The C_3 has a P'_1 at the discrete P_8, meets the C_6 again in $4P'_1$ and meets the g_2. Each curve is of genus 1.

8. $d_8 + g_2 + C^2_5 + C^2_4$; $(d_2 + g_1) + g_2 + C^2_5 + C^2_4$. The C_5 has a P'_2 at the discrete P_8 and $3P'_1$ on the g_2. The C_4 has P'_1 at the $3P_3$ and meets the C_5 again in $4P'_1$. Each curve is of genus 1.

9. $d_8 + g_2 + C^2_4 + C^2_3 + C^2_2$. The C_4 and C_2 each have P'_1 at the $3P_8$ and meet in another P'_1. The C_3 has a P'_1 at the discrete P_8, meets the C_2 once and the C_4 thrice again and meets the g_2 once. $p' = 1$ for both the C_4 and the C_8.

10. $(\delta_{2,1} + g_1) + g_2 + C^2_8$. The C_8 has P'_2 at the $3P_8$. $p' = 4$, 3, or 2.

11. $(\delta_{2,1} + g_1) + g_2 + 2C^2_4$. Each C_4 has P'_1 at the $3P_8$ and meets the other C_4 in $3P'_1$. $p' = 1$ for each C_4.

12. $d_8 + (2g_2) + C^2_8$. The C_8 has a P'_8 at the discrete P_8 and $4P'_1$ on the $(2g_2)$. $p' = 4$, 3 or 2.

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* See Wiman, *loc. cit.*, pp. 58 and 59. Snyder, *Amer. Jour. of Math.*, Vol. 25, p. 365; Vol. 27, p. 102.

13. $d_3 + (2g_2) + C_6^2 + C_2^2$. The C_6 has a P_2' at the discrete P_3 and $2P_1'$ on the $(2g_2)$. $p' = 3$ or 2. The C_2 has P_1' at the $3P_3$ and meets the C_6 again in $2P_1'$.

14. $d_3 + (2g_2) + C_5^2 + C_3^2$. The C_5 has a P_2' at the discrete P_3 and $3P_1'$ on the g_2. The C_3 has a P_1' at the discrete P_3 and meets the C_5 again in $3P_1'$. It meets the $(2g_2)$ once.

15. $d_3 + (2g_2) + 2C_3^2 + C_2^2$. Each C_3 has a P_1' at the discrete P_3, touches the torsal plane of the $(2g_2)$ at a point on the $(2g_2)$ and has two P_1' on the other g_2. $p' = 1$ for each C_3. The C_2 has P_1' at the $3P_3$ and meets each C_3 again in a P_1'.

16. $(\delta_{2,1} + g_1) + (2g_2) + C_7^2$. The C_7 has a P_2' at the P_3 on the directrix and $3P_1'$ on the $(2g_2)$. $p' = 4$, 3, or 2.

17. $(\delta_{2,1} + g_1) + 2g_2 + C_4^2 + C_3^2$. The C_4 has P_1' at the $3P_3$. The C_3 has a P_1' at the P_3 on the directrix and a P_1' on the $(2g_2)$. It meets the C_4 again in $3P_1'$. Each curve is a genus 1.

$$p = 3.$$

108. Take a generator through the single P_3' for $y = z = 0$ in transformation II. The R_7 may thus be transformed into a C_6 of genus 3, with $2P_1'$ on $\bar{x} = \bar{y} = 0$ and, on $\bar{y} = \bar{z} = 0$, a P_1', and a P_2' at which both tangents are not coplanar with $\bar{y} = \bar{z} = 0$. The scroll of bisecants is an R_{11} having $\bar{x} = \bar{y} = 0$ for d_5. Its genus is, by formula,* seven, but may reduce to three since $\bar{x} = \bar{y} = 0$ may be the intersection of four planes of the double developable. Since the C_6 lies on a quadric, the R_{11} cannot have a component R_2, K_2 or R_3. It may, however, break up into an R_8 of genus 3 or 2 and a K_3 of genus 1 or into an R_7 and an R_4, either pair of forms corresponding to a C_6^2 and a C_3^2 on the R_7. The R_{11} cannot break up into an R_5 and an R_6 for one of them would have to have the C_6 as double curve. No such R_5 or R_6 exists.

109. From considerations similar to the above it is readily seen that the double curve of the R_7 can break up into three components, if at all, only as $3C_3$ each of genus 1. That it can so decompose is readily seen by Salmon's generation of scrolls as the locus of a line meeting three curves. Take for the three curves the directrix and $2C_3$ of genus 1 each meeting the directrix and

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[*] See paragraph 9.

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having three common points at two of which the tangents to each curve meet the directrix. The R_{18} thus formed, consists of the plane pencils to the intersections of the two curves and the directrix, two of which count twice, the perspective cones of each C_3 from the intersection of the other C_3 with the directrix, and an R_7 with a d_3 having the $2C_3$ for double curves. If the directrix is taken so as not to be coplanar with any other pair of tangents of the cubics, then the R_7 must be of genus 3 for no such R_7 of lower genus has two such C_3^2. The scroll of bisecants of the C_6 may, therefore, break up into $2R_4$ and a K_8 each of genus 1. The C_6 may pass through $\bar{x}=\bar{y}=\bar{z}=0$.

1. $d_3+C_9^2$; $(d_2+g_1)+C_9^2$. The C_9 has a P_8' at the P_8. $p'=7, 6, 5, 4$, or 3.

2. $d_3+C_6^2+C_3^2$; $(d_2+g_1)+C_6^2+C_3^2$. The C_6 has a P_2' at the P_8. $p'=3$ or 2. The C_3 has a P_1' at the P_8 and meets the C_7 in $4P_1'$. Its genus is 1.

3. $d_3+3C_3^2$; $(d_2+g_1)+3C_3^2$. Each C_3 has a P_1' at the P_3 and meets each of the other C_3 in $2P_1'$. Each is of genus 1.

110. When the R_7 has a double contact directrix the P_8 is on the directrix. The R_7 is transformed by II into a C_6 with a P_2' at $\bar{x}=\bar{y}=\bar{z}=0$ at which one branch touches $y=0$. The C_6 lies on a K_3 having $\bar{y}=\bar{z}=0$ as a generator. The residual scroll of bisecants is an R_{10} having $y=z=0$ for double generator. Its genus is, in general, six but may reduce to three since $x=y=0$ may lie in three planes of the double developable of the C_6. The R_{10} may break up into an R_7 of genus 2 and a K_3 of genus 1 or into an R_6 of genus 2 and an R_4 of genus 1.

4. $(\delta_{2,1}+g_1)+C_8^2$. The C_8 has a P_2' at the P_3. $p'=6, 5, 4$, or 3.

5. $(\delta_{2,1}+g_1)+C_5^2+C_3^2$. The C_5 and C_3 each have a P_1' at the P_8 and meet in four other P_1'. The C_5 is of genus 2, the C_3 of genus 1.

Directrix a Four-fold Line on the Surface.

$$p=0$$

Triple Curves or Triple Generator.

111. When the R_7 has a C_8^3, the surface transforms into a C_4 having $x=y=0$ as directrix of the R_2 of trisicants on which the C_4 lies. It meets $\bar{x}=\bar{y}=0$ once. $\bar{z}=\bar{w}=0$ meets the C_4 in 0, 1, or 2 points.

112. When the R_7 has a C_2^3, the surface transforms into a plane C_4 with one point on $\bar{x}=\bar{y}=0$. It has $3P_2'$ or a P_3'. $\bar{z}=\bar{w}=0$ either does not meet the curve or meets it in a simple, a double, or a triple point.

113. In general, when the R_7 has a g_3, it transforms by II into a C_4 of the second kind having a P_1' on $\bar{x}=\bar{y}=0$. The scroll of bisecants is an R_6. When the R_7 has also a g_2, the C_4 has a P_2'. The scroll of bisecants is an R_5. The C_4 may pass through $\bar{x}=\bar{y}=\bar{z}=0$.

114. When the g_3 coincides with $x=y=0$, take the discrete P_3 and a P_2 as fundamental points in transformation I. The R_7 goes into a C_5 with $2P_1'$ on $\bar{x}=\bar{y}=0$. It passes through one fundamental point and has a P_3' on $\bar{z}=\bar{w}=0$. The scroll of bisecants is an R_5.

1. $d_4+C_3^3$; $(d_3+g_1)+C_3^3$; $(d_2+2g_1)+C_3^3$.
2. $d_4+3g_2+C_2^3$; $(d_3+g_1)+3g_2+C_2^3$; $(d_2+g_2)+2g_2+C_2^3$.
3. $d_4+g_3+C_2^3$; $(d_3+g_1)+g_3+C_2^3$; $(d_1+g_3)+C_2^3$.
4. $d_4+g_3+C_6^2$; $(d_3+g_1)+g_3+C_6^2$; $(d_1+g_3)+C_6^2$. The C_6 has a P_3' at the P_3 and $3P_1'$ on the g_3. $p'=0$.
5. $d_4+g_3+g_2+C_5^2$; $(d_3+g_1)+g_3+g_2+C_5^2$. The C_5 has a P_2' and a P_1' on the g_2 and $3P_1'$ on the g_3. $p'=0$.

Double Curves.

115. Taking $2P_3'$ for the fundamental points in transformation I, the R_7 transforms into a C_4 with a P_1' on $\bar{x}=\bar{y}=0$. The scroll of bisecants is an R_6. It cannot break up. $\bar{z}=\bar{w}=0$ meets the C_4 in 0, 1, 2 or 3 points.

116. When the R_7 has a g_2 the C_4 has a P_2'. The scroll of bisecants is an R_5. $\bar{z}=\bar{w}=0$ meets the C_4 in 0, 1, or 2 P_1' or in the P_2' and in 0 or 1 P_1'. When one fundamental point lies on the R_5 of bisecants, the R_7 has $2g_2$. When both fundamental points lie on the C_5 then the R_7 has $3g_2$.

1. $d_4+C_9^2$; $(d_3+g_1)+C_9^2$; $(d_2+2g_1)+C_9^2$; $(d_1+3g_1)+C_9^2$. The C_9 has P_3' at the $3P_3$. $p'=0$.
2. $d_4+g_2+C_8^2$; $(d_3+g_1)+g_2+C_8^2$; $(d_2+2g_1)+g_2+C_8^2$; $(d_2+g_2)+C_8^2$; $(d_1+g_1+g_2)+C_8^2$. The C_8 has P_3' at the two discrete P_3 and a P_2' on the g_2. $p'=0$.
3. $d_4+2g_2+C_7^2$; $(d_3+g_1)+2g_2+C_7^2$; $(d_2+2g_1)+2g_2+C_7^2$; $(d_2+g_2)+g_2+C_7^2$. The C_7 has a P_3' at the discrete P_3 and a P_2' and a P_1' on each g_2. $p'=0$.
4. $d_4+3g_2+C_6^2$; $(d_3+g_1)+3g_2+C_6^2$; $(d_2+g_2)+2g_2+C_6^2$. The C_6 has a P_2' and a P_1' on each g_2. $p'=0$.

$p = 1$

Triple Curve or Triple Generator.

117. The R_7 cannot have a C^3_3 since a C^3_3 and a d_4 would give a nodal curve of order 15. When the R_7 has a C^3_2 it transforms by I into a plane C_4 with $2P'_2$. $\bar{z} = \bar{w} = 0$ either does not meet the curve or meets it in a P'_1 or a P'_2.

118. When the R_7 has a g_3 it transforms by II into a gauche C_4 of the first kind meeting $\bar{x} = \bar{y} = 0$ in a P'_1. The scroll of bisecants is an R_5. The C_4 may pass through $\bar{x} = \bar{y} = \bar{z} = 0$. The g_3 cannot coincide with the directrix of the R_7 for the latter would then be a simple rectilinear directrix which a scroll of genus one cannot have.

1. $d_4 + 2g_2 + C^3_2$; $(d_3 + g_1) + 2g_2 + C^3_2$; $(d_2 + g_2) + g_2 + C^3_2$.
2. $d_4 + g_3 + C^2_5$; $(d_3 + g_1) + g_3 + C^2_3$. The C_5 has $3P'_1$ on the g_3. $p' = 1$.

Double Curves.

119. Taking the $2P_3$ for fundamental points, the R_7 transforms into a C_4 of the first kind. The scroll of bisecants is an R_5. $\bar{z} = \bar{w} = 0$ meets the C_4 in 0, 1 or 2 P'_1.

120. When one fundamental point lies on the R_5 of bisecants, the R_7 has a g_2. When both fundamental points lie on the R_5 the R_7 has $2g_2$. By taking one fundamental point at a P_2 instead of at a P_3 of the R_7 it may be seen that one g_2 may coincide with the directrix.

1. $d_4 + C^2_8$; $(d_3 + g_1) + C^2_8$; $(d_2 + 2g_1) + C^2_8$. The C_8 has P'_3 at the $2P_3$. $p' = 1$.
2. $d_4 + g_2 + C^2_7$; $(d_3 + g_1) + g_2 + C^2_7$; $(d_2 + 2g_1) + g_2 + C^2_7$; $(d_2 + g_2) + C^2_7$. The C_7 has a P'_3 at the discrete P_3 and a P'_2 and a P'_1 on the g_2. $p' = 1$.
3. $d_4 + 2g_2 + C^2_6$; $(d_3 + g_1) + 2g_2 + C^2_6$; $(d_2 + g_2) + g_2 + C^2_6$. The C_6 has a P'_2 and a P'_1 on each g_2. $p' = 1$.

$p = 2$

Triple Curve.

121. The R_7 can have only a triple conic. By I it transforms into a plane C_4 with a P'_2. $\bar{z} = w = 0$ either does not meet it or meets it in a P'_1 or a P'_2.

1. $d_4 + g_2 + C^3_2$; $(d_3 + g_1) + g_2 + C^3_2$; $(d_2 + g_2) + C^3_2$.

Double Curves.

122. The R_7 has just one P_3 and transforms into a C_5 meeting $\bar{x}=\bar{y}=0$ twice and passing through one fundamental point. The scroll of bisecants is an R_6. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, 2 points besides the fundamental point.

123. When the fundamental point not on the C_5 lies on the R_6 of bisecants the R_7 has a g_2. It may, as before, be shown that the g_2 may coincide with the directrix.

1. $d_4+C_7^2$; $(d_3+g_1)+C_7^2$; $(d_2+2g_1)+C_7^3$. The C_7 has a P_3' at the P_3. $P'=2$.

2. $d_4+g_2+C_6^2$; $(d_3+g_1)+g_2+C_6^2$; $(d_2+g_2)+C_6^2$. The C_6 has a P_2' and a P_1' on the g_2. $p'=2$.

$$p=3.$$

Triple Conic.

124. The R_7 transforms into a non-singular plane C_4. $\bar{z}=\bar{w}=0$ meets the curve in 0 or 1 P_1'.

1. $d_4+C_2^3$; $(d_3+g_1)+C_2^3$.

Double Curve.

125. The R_7 has no P_3. It transforms by I into a C_6 meeting $\bar{x}=\bar{y}=0$ thrice and passing through each fundamental point. The scroll of bisecants is an R_7. $\bar{z}=\bar{w}=0$ meets the C_6 in 0 or 1 points besides the fundamental point.

1. $d_4+C_6^2$; $(d_3+g_1)+C_6^2$. The C_6 is of genus 3.

Directrix a Fivefold Line on the Surface.

$$p=0.$$

126. The R_7 transforms by I into a C_5 meeting $\bar{x}=\bar{y}=0$ thrice. The scroll of bisecants is an R_4. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, 2, 3 or 4 points.

127. When the R_7 has a g_2 it transforms into a C_5 with a P_2'. The scroll of bisecants is an R_3. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, 2 or 3 P_1' or in the P_2' and 0, 1 or 2 P_1'.

128. From this configuration it is readily seen that the R_7 may have a contact directrix $(\delta_{2,2}+g_1)$ for such an R_7 is obtained when $\bar{z}=\bar{w}=0$ is taken

to coincide with the simple directrix of the R_3 of bisecants. Since both fundamental points lie on the R_3 of bisecants such an R_7 has $3g_2$.

129. When the R_7 has $2g_2$, it transforms into a C_5 with $2P'_2$. The scroll of bisecants is an R_2 or a K_2. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, or 2 P'_1 or in a P'_2 and 0 or 1 P'_1 or in $2P'_2$.

130. When the R_7 has $3g_2$ — and no $(\delta_{2,2}+g_1)$ — it transforms into a plane C_5 with a P'_3 on $\bar{x}=\bar{y}=0$ and $3P'_2$. $\bar{z}=\bar{w}=0$ meets the C_5 in 0 or 1 P'_1 or in a P'_2.

1. $d_5+C_5^2$; $(d_4+g_1)+C_5^2$; $(d_3+2g_1)+C_5^2$; $(d_2+3g_1)+C_5^2$; $(d_1+4g_1)+C_5^2$.

2. $d_5+g_2+C_4^2$; $(d_4+g_1)+g_2+C_4^2$; $(d_3+2g_1)+g_2+C_4^2$; $(d_2+3g_1)+g_2+C_4^2$; $(d_3+g_2)+C_4^2$; $(d_2+g_1+g_2)+C_4^2$; $(d_1+2g_1+g_2)+C_4^2$. The C_4 has a P'_1 on the g_2.

3. $d_5+2g_2+C_3^2$; $(d_4+g_1)+2g_2+C_3^2$; $(d_3+2g_1)+2g_2+C_3^2$; $(d_3+g_2)+g_2+C_3^2$; $(d_2+g_1+g_2)+g_2+C_3^2$; $(d_1+2g_2)+C_3^2$. The C_3 may be either plane or gauche. It has a P'_1 on each g_2.

4. $d_5+3g_2+C_2^2$; $(d_4+g_1)+3g_2+C_2^2$; $(d_3+g_2)+2g_2+C_2^2$. The C_2 has a P'_1 on each g_2.

5. $(\delta_{2,2}+g_1)+3g_2$. The two generators in an arbitrary plane through the directrix intersect on the directrix.

$$p=1$$

131. The R_7 transforms by I into a C_5 of genus 1 having $3P'_1$ on $\bar{x}=\bar{y}=0$. The scroll of bisecants is an R_3. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, 2 or 3 P'_1.

132. Taking the simple directrix of the R_3 of bisecants for $\bar{z}=\bar{w}=0$ we obtain a contact directrix. Since each fundamental point lies on the scroll of bisecants, the R_7 has $2g_2$.

133. When the R_7 has a g_2, the C_5 has a P'_2. The scroll of bisecants is an R_2 or a K_2. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1 or 2 P'_1 or in the P'_2 and 0 or 1 P'_1.

133. When the R_7 has $2g_2$ — and no $(\delta_{2,2}+g_1)$ — it transforms into a plane C_5 with a P'_3 on $\bar{x}=\bar{y}=0$ and $2P'_2$. $\bar{z}=\bar{w}=0$ meets the C_5 in 0 or 1 P'_1 or in a P'_2.

1. $d_5+C_4^2$; $(d_4+g_1)+C_4^2$; $(d_3+2g_1)+C_4^2$; $(d_2+3g_1)+C_4^2$.

2. $d_5 + g_2 + C_3^2$; $(d_4 + g_1) + g_2 + C_3^2$; $(d_3 + 2g_1) + g_2 + C_3^2$; $(d_3 + g_2) + C_3^2$; $(d_2 + g_1 + g_2) + C_3^2$. The C_3 is either plane or gauche. It has a P_1' on the g_2.

3. $d_5 + 2g_2 + C_2^2$; $(d_4 + g_1) + 2g_2 + C_2^2$; $(d_3 + g_1) + g_2 + C_2^2$. The C_2 has a P_1' on each g_2.

4. $(\delta_{2,2} + g_1) + 2g_2$. The two generators in an arbitrary plane through the directrix intersect on the directrix.

$$p = 2.$$

135. The R_7 transforms into a C_5 of genus two. The scroll of bisecants is an R_2 or a K_2. $\bar{z} = \bar{w} = 0$ meets the C_5 in 0, 1 or 2 P_1'.

136. When the R_7 has a g_2 — and no $(\delta_{2,2} + g_1)$ —, it transforms into a a plane C_5 with a P_3' on $\bar{x} = \bar{y} = 0$ and a P_2'. $\bar{z} = \bar{w} = 0$ meets the C_5 in 0 or 1 P_1' or in a P_2'.

137. When the R_7 has a contact directrix it has a g_2 and may therefore be transformed by II into a C_5 lying on K_2 which has its vertex at $\bar{x} = \bar{y} = \bar{z} = 0$ and has $\bar{x} = \bar{y} = 0$ for generator.

1. $d_5 + C_3^2$; $(d_4 + g_1) + C_3^2$; $(d_3 + 2g_1) + C_3^2$. The C_3 may be either plane or gauche.

2. $d_5 + g_2 + C_2^2$; $(d_4 + g_1) + g_2 + C_2^2$; $(d_3 + g_2) + C_2^2$. The C_2 has a P_1' on the g_2.

3. $(\delta_{2,2} + g_1) + g_2$. The two generators in an arbitrary plane through the directric intersect on the directrix.

$$p = 3.$$

138. When the R_7 does not have a contact directrix, it transforms into a plane C_5 with a P_3' on $\bar{x} = \bar{y} = 0$. $\bar{z} = \bar{w} = 0$ meets the C_5 in 0 or 1 P_1'.

139. When the R_7 has a contact directrix, it may be transformed by II into a C_6 having $\bar{x} = \bar{y} = \bar{z} = 0$ a P_2' with one branch tangent to $y = 0$ and lying on a K_2 which has $\bar{x} = \bar{y} = 0$ for generator.

1. $d_5 + C_2$; $(d_4 + g_1) + C_2$.

2. $(\delta_{2,2} + g_1)$. The two generators in an arbitrary plane through the directrix intersect on the directrix.

Directrix a Six-fold Line on the Surface.

140. The R_7 is necessarily unicursal. It transforms by I into a C_6 meeting $\bar{x}=\bar{y}=0$ five times. $\bar{z}=\bar{w}=0$ meets the C_6 in 0, 1, 2, 3, or 4 P_1'. If $\bar{z}=\bar{w}=0$ meets it in $5P_1'$ then R_7 belongs to a special linear congruence and has already been enumerated.

1. d_6; (d_5+g_1); (d_4+2g_1); (d_3+3g_1); (d_2+4g_1).

CORNELL UNIVERSITY, *July* 28, 1905.

Beiträge zur Nicht-Euklidischen Geometrie. I. II. III.

VON E. STUDY.

Unter gemeinsamem Titel denkt der Verfasser eine Reihe von Untersuchungen über die algebraische und Differentialgeometrie im Nicht-Euklidischen Raume herauszugeben, einen Gegenstand, der ihn schon seit Jahren beschäftigt hat, ohne dass es doch zu einer mehr als skizzenhaften Veröffentlichung gekommen wäre.

In dem ersten der vorliegenden drei Beiträge wird eine anscheinend neue Eigenschaft der geraden Linie erörtert. Die zweite Abhandlung ist von der ersten unabhängig. Sie ist überschrieben: Die Begriffe links und rechts*), es werden aber darin auch andere Grundbegriffe der Nicht-Euklidischen Geometrie behandelt. Sie ist als Einleitung zu weiteren Arbeiten gedacht, und deshalb elementar gehalten. Sie kann vielleicht auch zur Einführung in die Methoden dienen, die der Verfasser in seiner Geometrie der Dynamen (Leipzig, 1903) angewendet hat. Die dritte Mittheilung enthält eine einfache Anwendung auf die Schraubenflächen im Nicht-Euklidischen Raume positiver Krümmung, die als geodätische Linien in einer vierfach ausgedehnten Mannigfaltigkeit betrachtet werden.

I.

Gerade und Punkt als Extreme.

Es scheint noch nicht beachtet worden zu sein, dass der Satz von der Geraden als kürzestem Weg zwischen zwei Punkten in der hyperbolischen Geometrie zwei Analoga hat, bei denen *der Punkt* bemerkenswerther Weise nicht als kürzester, sondern als *längster Weg zwischen zwei Geraden* erscheint.

Ergänzen wir in bekannter Weise das Punktcontinuum der *Lobatschewsky*'schen Geometrie durch Zufügung unendlich ferner und sogenannter idealer

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*) Bei einer anderen Gelegenheit denken wir noch zu zeigen, dass ein Theil vom Inhalte der Begriffe links und rechts zu seiner Entwickelung den Maassbegriff nicht erfordert, sondern der reellen projectiven Geometrie des Raumes angehört.

Punkte zum Punktcontinuum der projectiven Geometrie, so kann man diese natürlich noch präcise zu fassende Behauptung auch in der correlativen Form aussprechen: *Die Gerade oder vielmehr ein Stück einer Geraden erscheint als längster Weg zwischen zwei Punkten.*

Wir betrachten zunächst die ebene Geometrie, vom Krümmungsmaasse — 1, gehörig zu dem absoluten Polarsystem

$$(xy) = x_1y_1 - x_2y_2 - x_3y_3 = 0. \tag{1}$$

Wir können dann die — hier stets als *positiv* $\{\geqq 0\}$ betrachtete — Entfernung zweier Punkte im Inneren $\{(xx) > 0,\ (yy) > 0\}$ des absoluten Kegelschnittes eindeutig erklären durch die Formel

$$(x, y) = \text{arc cos}\, h \sqrt{\frac{(x\,y)^2}{(x\,x)\ (y\,y)}}, \tag{2}$$

wofern wir dem Wurzelzeichen und der Function arc cos h den positiven Werth beilegen. Dieselbe Formel liefert aber eine reelle „Entfernung" der Punkte x, y auch noch in dem Falle, wo diese auserhalb des absoluten Kegelschnittes und zwar so gelegen sind, dass ihre gerade Verbindungslinie den absoluten Kegelschnitt trifft. Ueberdies besteht auch in diesem Falle für drei Punkte y, x, z in gerader Linie, deren mittlerer x die beiden anderen trennt, die Gleichung

$$(y, z) = (y, x) + (x, z). \tag{3}$$

Wir wollen nun ein Dreieck x, y, z von verschiedenen ausserhalb gelegenen Punkten so annehmen, dass die Verbindungslinie von je zweien den absoluten Kegelschnitt trifft; derart also, dass

$$\begin{aligned} &(yz)^2 - (yy)\,(zz) \geqq 0, \quad (xx) < 0, \\ &(zx)^2 - (zz)\,(xx) \geqq 0, \quad (yy) < 0, \\ &(xy)^2 - (xx)\,(yy) \geqq 0, \quad (zz) < 0. \end{aligned} \tag{4}$$

Wir können dann zwei Familien solcher Dreiecke unterscheiden:

Ist

$$(yz)\,(zx)\,(xy) < 0, \tag{5a}$$

so ist die grösste Seite der Dreiecks x, y, z immer grösser als die Summe der beiden anderen, oder mindestens dieser Summe gleich. Und zwar tritt der Fall der Gleichheit dann und nur dann ein, wenn die drei Punkte in gerader Linie liegen.

Ist dagegen

$$(yz)\,(zx)\,(xy) > 0, \tag{5b}$$

so kann die grösste Seite grösser oder kleiner als die Summe der beiden anderen sein, oder auch dieser gleich. Niemals aber liegen dann die drei Punkte in gerader Linie.

Nehmen wir an, es sei

$$(y, z) \geqq (y, x), \quad (y, z) \geqq (x, z), \tag{6}$$

und fragen wir, unter welchen Veraussetzungen wir schliessen können, dass

$$(y, z) \geqq (y, x) + (x, z) \tag{7a}$$

sein muss.

Die Ungleichung (7a) können wir offenbar auch so schreiben:

$$\cos h \{(y, z) - (y, x)\} \geqq \cos h\, (x, z).$$

Lösen wir hier mit Benutzung des Additionstheorems den hyperbolischen Cosinus links auf, so nimmt diese Ungleichung, bei positiven Werthen aller vorkommenden Wurzelgrössen, die Form an:

$$\sqrt{(yz)^2}\,\sqrt{(xy)^2} + (yy)\,\sqrt{(xz)^2} \geqq \sqrt{(yz)^2 - (yy)\,(zz)}\,\sqrt{(xy)^2 - (xx)\,(yy)}.$$

Hier sind nun nach Voraussetzung (4) und (6) beide Seiten positive Grössen. Völlig äquivalent mit der letzten Ungleichung ist also die aus ihr durch Quadriren entstehende:

$$\begin{aligned}(xx)\,(yy)\,(zz) - 2\sqrt{(yz)^2}\,\sqrt{(zx)^2}\,\sqrt{(xy)^2} - \\ - (xx)\,(yz)^2 - (yy)\,(zx)^2 - (zz)\,(xy)^2 \geqq 0.\end{aligned}$$

Anderseits ist

$$(xx)\,(yy)\,(zz) + 2\,(yz)\,(zx)\,(xy) - (xx)\,(yz)^2 - (yy)\,(zx)^2 - (zz)\,(xy)^2 \geqq 0,$$

weil die linke Seite das Quadrat der Determinante (xyz) darstellt. Wenn also die Ungleichung (5a) stattfindet, so fallen die beiden letzten Ungleichungen zusammen, und es ergiebt sich (7a). Sie bestehen beide als Gleichungen, wenn $(xyz) = 0$ ist. Der umgekehrte Schluss aber, dass aus (7a) auch (5a) folgen müsste, kann nicht gezogen werden.

Nehmen wir jetzt an, dass (5b) stattfindet, so wird behauptet, dass

$$(y, z) \mathrel{\substack{>\\=\\<}} (y, x) + (x, z) \tag{7b}$$

sein kann, oder, nach Obigem, dass (bei Annahme der Ungleichungen (6) *entsprechend*)

$$(xyz)^2 - 4\,(yz)\,(zx)\,(xy) \mathrel{\substack{>\\=\\<}} 0$$

sein kann. In der That überzeugt man sich hiervon sehr leicht durch Beispiele, oder durch eine sogleich vorzuführende Ueberlegung. Dass (xyz) in diesem Falle nicht verschwinden kann, ergiebt sich daraus, dass dann in (7b) das Gleichheitszeichen gelten müsste. Dann müsste also — da $(xyz) = 0$ — das Product $(yz)(zx)(xy)$ verschwinden, was nach (4) unmöglich ist.

Um eine deutliche Vorstellung von den zugehörigen Dreiecksfiguren zu erhalten, wollen wir die Punkte y, z als gegeben annehmen. Man sieht leicht, dass dann die Gesammtheit der Punkte x, die den Ungleichungen (5a) oder (5b) genügen, dargestellt wird durch die in der Figur (1) schraffirten Gebiete.

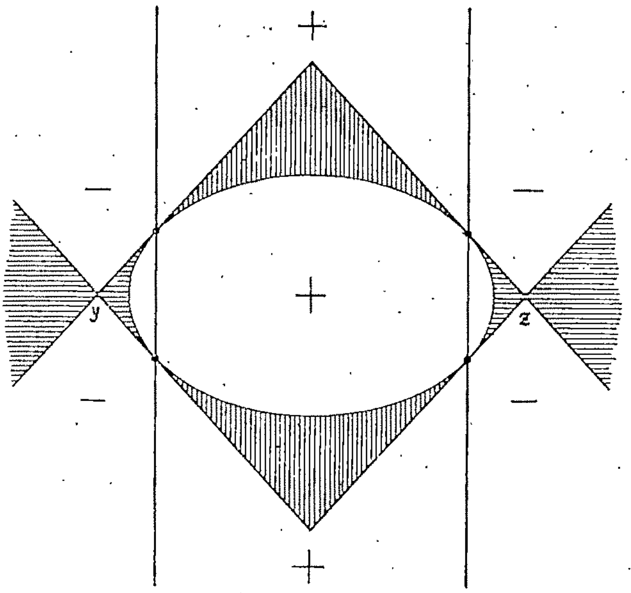

FIG. 1.

Verticale Schraffirung besagt, dass

$$(yz)\,(zx)\,(xy) > 0,$$

horizontale, dass

$$(yz)\,(zx)\,(xy) < 0$$

ist.

Figur (2) zeigt sodann ebenfalls durch verticale und horizontale Schraffen an, wo

$$(y,\,z) - (y,\,x) - (x,\,z) \geqq 0, \quad \text{oder} \quad (y,\,z) - (y,\,x) - (x,\,z) \leqq 0 \ \text{ist.}$$

Diese Figur zeigt ausserdem, stark ausgezogen, den Ort aller Punkte x, für die

$$(y,\,z) - (y,\,x) - (x,\,y) = 0$$

ist. *Dieser Ort besteht aus drei völlig getrennten Linienzügen,* deren einer gerade ist, während die beiden anderen der Curve 2. Ordnung

$$(yzx)^2 - 4\,(yz)\,(yx)\,(zx) = 0 \tag{8}$$

angehören. Diese Curve berührt zunächst die Polaren der Punkte y, z, die Geraden $(yx) = 0$ und $(zx) = 0$, in ihren Schnittpunkten mit $(yzx) = 0$. Bildet man sodann, unter Verwendung der Bezeichnungen

$$(ux) = u_1 x_1 + u_2 x_2 + u_3 x_3, \quad (uv) = u_1 v_1 - u_2 v_2 - u_3 v_3$$

ihre Gleichung in Liniencoordinaten, so findet sich, nach Abspaltung des nicht

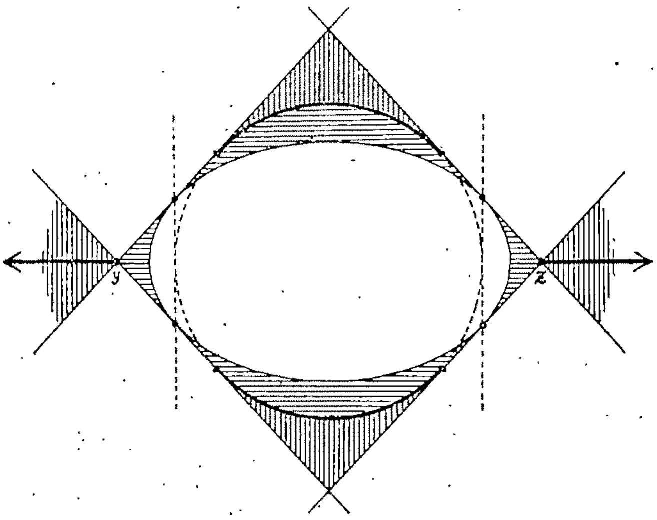

FIG. 2.

verschwindenden Factors $4\,(yz)\,\{(yz)^2 - (yy)\,(zz)\}$ von der quadratischen Covariante des Ausdrucks (8), die Gleichung

$$(yz)\,(uu) - (uy)\,(uz) = 0\,. \tag{9}$$

Man sieht also, dass die Curve (8) die Minimalgeraden durch y, z — die von y, z an den absoluten Kegelschnitt gehenden Tangenten — berührt, natürlich in reellen Punkten. Schliesslich geht durch die beiden Berührungspunkte der Curve (8) mit den von z ausgehenden Minimalgeraden der in der Figur nicht gezeichnete, aber leicht zu ergänzende Ort $(y, x) = (y, z)$. Die Gebiete innerhalb der Curven $(y, x) = (y, z)$ und $(z, x) = (y, z)$ gehören dann zu den Ungleichungen $(y, z) < (y, x)$ und $(y, z) < (z, x)$, werden also horizontal zu schraffiren sein. Die Curve (8) aber wird, soweit sie *ausserhalb* dieser Gebiete verläuft, die beiden Arten der Schraffirung trennen. Man sieht nun ohne Mühe, dass die Schraffen richtig eingetragen sind, wenn man x auf den Randcurven der verschiedenen Gebiete laufen lässt.

Zu grösserer Deutlichkeit fügen wir noch — in collinearer Abänderung — eine dritte Figur hier an, die zeigt, wie bei gegebenem x die Punkte y und z

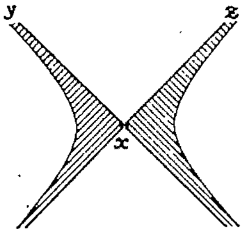

Fig. 8.

liegen müssen, wenn gleichzeitig

$$(yz)\,(zx)\,(xy) < 0, \quad (yz) \geqq (y,\, x) + (x,\, z) \tag{10}$$

sein soll.

In diesem Falle, der nun noch näher betrachtet werden soll, werden wir sagen: „*Der Punkt x trennt die Punkte y und z,*“ oder auch: „*Der Punkt x liegt zwischen y und z.*“ Wenn unter den Verbindungslinien der Punkte x, y, z keine Minimalgerade vorkommt, haben dann die Polaren der Punkte x, y, z eine solche gegenseitige Lage, wie sie durch Figur 4*a*) angezeigt wird; während die Dreiecke der zweiten Familie der Figur 4*b*) entsprechen.

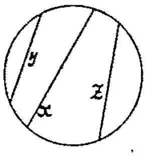

Fig. 4a.

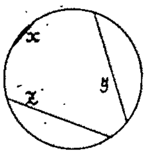

Fig. 4b.

Um nicht zu sehr ins Breite gehen zu müssen, entnehmen wir nunmehr der Anschauung (Fig. 4a), oder einer nahe liegenden Continuitätsbetrachtung, die folgenden Sätze:

Liegt der Punkt x' zwischen y und z, und der Punkt x'' zwischen (y und x' oder) x' und z, so liegt x'' auch zwischen y und z.

Liegen zwei verschiedene Punkte zwischen den Punkten y, z, und zwar so, dass *ihre gerade Verbindungslinie den absoluten Kegelschnitt trifft, so haben alle vier Punkte eine bestimmte Anordnung* y, x', x'', z *oder* $z, x'', x' y$.

Es liegt nämlich nach geeigneter Wahl der Bezeichnung x' *zwischen* y *und* x'', *und* x'' *zwischen* x' *und* z, *aber nicht* x'' *zwischen* y *und* x', *und nicht* x' *zwischen* x'' *und* z.

Liegt ferner der Punkt x *zwischen zwei solchen Punkten* x' *und* x'', *so liegt* x *auch zwischen* y *und* z.*)

Die Art, wie diese Sätze zu Stande kommen, wird wohl hinreichend durch die beiden Figuren 5 erläutert. Man schliesst daraus ohne Mühe den folgenden bemerkenswerthen Satz:

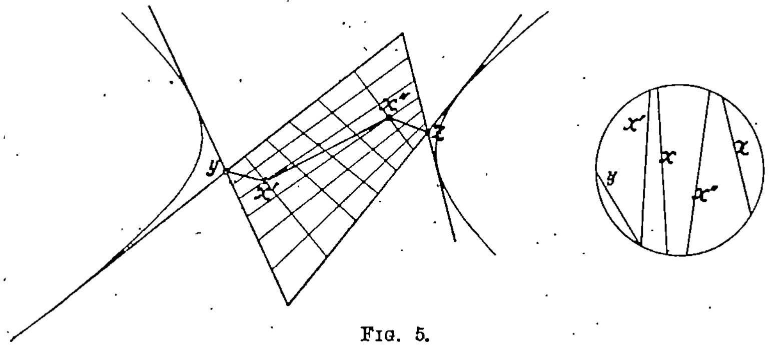

FIG. 5.

Es sei vorgelegt eine Menge von Punkten ausserhalb des absoluten Kegelschnittes, die ein CONTINUUM *bildet, und ausserdem so beschaffen ist, dass die gerade Verbindungslinie von je zwei verschiedenen Punkten der Menge den absoluten Kegelschnitt trifft.*

Jeder beliebige Punkt der Menge, mit Ausnahme von höchstens zweien y_0, z_0, *hat dann die folgende Eigenschaft: Nimmt man diesen Punkt* x *aus der Menge heraus, so bilden die übrigen Punkte der Menge zwei ebenfalls continuirliche Theilmengen* (y) *und* (z), *derart, dass* x *alle Punkte* y *der einen trennt von allen Punkten* z *der anderen.*

Alle Punkte der Menge zwischen zwei verschiedenen y, z *bilden nach Zufügung von* y *und* z *selbst einen abgeschlossenen continuirlichen Curvenzug, dessen Punkte* x *eindeutig-umkehrbar und stetig den Werthen eines Parameters* t *von Null bis Eins zugeordnet werden können.*

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*) Dieses Alles würde sich strenge, aber, soviel wir sehen, nicht ganz kurz, mit rein algebraischen Mitteln beweisen lassen. — Wo es sich nicht um Fundamentalsätze grosser Theorieen handelt, dürfen wohl auch abkürzende Darstellungsformen angewendet werden, wenn es nur möglich ist, die entstehenden Lücken im Bedarfsfalle auszufüllen.

Dieser Curvenzug hat eine völlig bestimmte Länge S, die den Ungleichungen

$$0 \leqq S \leqq (y, z)$$

genügt.

Ferner ergiebt sich: Sind zwei Ausnahmspunkte y_0, z_0 der bezeichneten Art vorhanden, so besteht die ganze Menge aus diesen Punkten selbst und einem sie verbindenden Curvenzug von bestimmter Bogenlänge, dessen sämmtliche von y_0, z_0 verschiedene Punkte zwischen y_0 und z_0 liegen. Die im Satze genannten Punkte y, z erhalten immer die Eigenschaft solcher Grenzpunkte y_0, z_0, wenn man alle von y, z verschiedenen Punkte der Menge unterdrückt, die *nicht* zwischen y und z liegen.

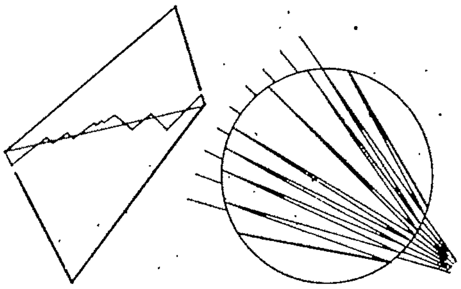

FIG. 6.

Nennen wir nun einen Curvenzug, wie er im Satze beschrieben ist, einen *Weg* zwischen y und z, so haben wir das Eingangs bezeichnete Resultat.

Unter allen Wegen zwischen zwei verschiedenen Punkten y, z, die ausserhalb des absoluten Kegelschnittes liegen, und deren gerade Verbindungslinie diesen trifft, ist der gerade Weg der längste.

Hierzu gehören als eine nützliche Ergänzung die weiteren Bemerkungen:

Ist die gerade Verbindungslinie der Punkte y, z eine Minimalgerade, d. h., berührt sie den absoluten Kegelschnitt, so giebt es nur den einen auf ihr selbst gelegenen Weg zwischen beiden Punkten.

In jedem anderen Falle giebt es unbegrenzt viele Wege zwischen y, z, und in beliebiger Nähe irgend eines von ihnen insbesondere noch unbegrenzt viele, die die Länge Null haben.

Die Figur 6 macht es anschaulich, wie ein Weg zu Stande kommt, der in beliebiger Nähe des längsten liegt, selbst aber die Länge Null hat.

Für die Punkte einer Geraden, die den absoluten Kegelschnitt *nicht* trifft, also ganz im idealen Gebiete verläuft, können wir eine elliptische Maassbestimmung festsetzen. Die Entfernung zweier Punkte x, y auf einer solchen Geraden wird dann nur bis aufs Vorzeichen und bis auf ein Multiplum einer gewissen Periode bestimmt sein, die gleich π angenommen werden darf. Wir wählen unter den verschiedenen Werthen dieser Function den absolut-kleinsten aus, betrachten ihn als eine positive Grösse, und bezeichnen diese mit (x, y). Indem wir für alle vorkommenden Wurzelgrössen auch hier deren positive Werthe setzen, haben wir dann die Gleichung

$$\cos(x, y) = \sqrt{\frac{(xy)^2}{(xx)\,(yy)}}, \tag{10}$$

und es ist immer

$$0 \leqq \cos(x, y) \leqq 1.$$

Dieselbe Formel (10) können wir dann auch noch in dem Grenzfall gelten lassen, wo die x, y enthaltende Gerade den absoluten Kegelschnitt *berührt*, aber weder x noch y in den Berührungspunkt fällt. Wir setzen dann $(x, y) = 0$.

Wir wollen nun zeigen, dass die so erklärte Entfernung (x, y), ebenso aber auch ihr Supplement

$$[x, y] = \pi - (x, y)$$

gleichfalls die Eigenschaft hat, ein *Maximum* zu sein unter den Längen gewisser noch zu definirender „Wege“ zwischen den beiden Punkten.

Wir betrachten wieder ein Dreieck x, y, z von verschiedenen idealen Punkten, deren gerade Verbindungslinien aber nun den absoluten Kegelschnitt *entweder gar nicht treffen oder höchstens berühren* sollen:

$$\begin{aligned} (yz)^2 - (yy)\,(zz) &\leqq 0, \quad (xx) < 0, \\ (zx)^2 - (zz)\,(xx) &\leqq 0, \quad (yy) < 0, \\ (xy)^2 - (xx)\,(yy) &\leqq 0, \quad (zz) < 0. \end{aligned} \tag{11}$$

Ohne eine Beschränkung einzuführen, dürfen wir dann annehmen, dass

$$\frac{\pi}{2} \geqq (y, z) \geqq (y, x) \geqq (x, z) \geqq 0 \tag{12}$$

sei. Die der Reihe nach hypothetisch anzunehmenden Ungleichungen

$$(y, z) \overset{>}{\underset{<}{=}} (y, x) + (x, z) \tag{13}$$

lassen sich dann so schreiben:

$$\cos(x, z) - \cos(y, z)\cos(y, x) \overset{>}{\underset{<}{=}} \sin(y, z)\,.\,\sin(y, x).$$

Da hier nach Voraussetzung beide Seiten positive Grössen sind, so erhält man durch Quadriren äquivalente Ungleichungen,

$$(xx)(yy)(zz) - 2\sqrt{(yz)^2(zx)^2(xy)^2} -$$
$$- (xx)(yz)^2 - (yy)(zx)^2 - (zz)(xy)^2 \gtreqless 0.$$

Wie zuvor sieht man jetzt, dass nur das obere Zeichen oder das Gleichheitszeichen möglich ist, wenn

$$(yz)(zx)(xy) \leqq 0, \tag{14a}$$

während im Falle

$$(yz)(zx)(xy) > 0 \tag{14b}$$

eine Entscheidung nicht getroffen werden kann.

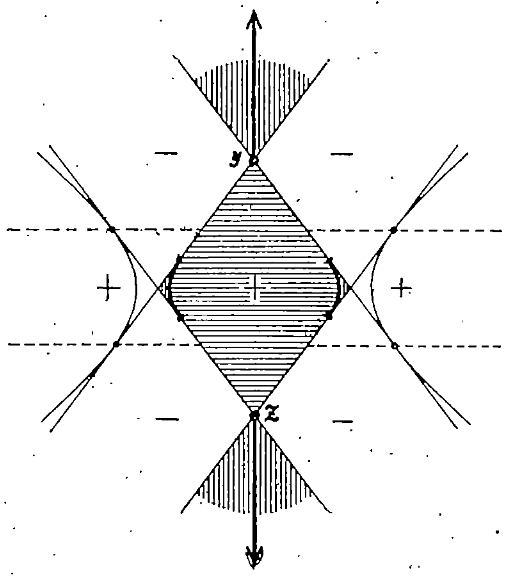

FIG. 7.

Es wird genügen, wenn wir die leicht zu behandelnden Grenzfälle

$$(y, z) = \frac{\pi}{2} = [y, z] \text{ und } (y, z) = 0, \ [y, z] = \pi$$

weiterhin zunächst ausschliessen und also annehmen, dass

$$0 < (y, z) < \frac{\pi}{2} < [y, z] < \pi$$

sei.

Die beiden Abschnitte, in die die Gerade yz durch die beiden Punkte y, z zerlegt wird, sind nun jeder Diagonale eines einfachen Vierseits, dessen Seiten Stücke von Minimalgeraden sind. (S. die Figur 7).

Im Inneren oder an der Grenze dieser beiden Vierseite liegen alle den Un- eichungen (11) genügenden Punkte x. Beide Vierseite haben im Uebrigen ver- hiedene Eigenschaften. Bei geradliniger Erstreckung des Integrals über das ogenelement zwischen y und z durch das Innere des einen Vierseits hindurch tsteht der kleinere Werth (y, z), bei Erstreckung durch das andere Vierseit der össere $[y, z]$. Das erste, wie wir der Bequemlichkeit halber sagen wollen, kleinere“ Vierseit liegt ganz im Gebiete $(yz)(zx)(xy) < 0$; das andere rössere“ wird durch die absoluten Polaren der Punkte y, z in drei Abschnitte rlegt, zwei äussere, in denen ebenfalls $(yz)(zx)(xy) < 0$ ist, und einen inneren, dem $(yz)(zx)(xy) > 0$ ist. Weiter erkennt man — ähnlich wie zuvor — *ss der Ort aller Punkte x, für die*

$$(y, z) = (y, x) + (x, z) \tag{15}$$

, *aus drei getrennten Curvenzügen besteht.* Der eine von diesen ist gerade, und rläuft ganz im kleineren Vierseit. Die beiden anderen liegen auf der irre- cibilen Curve 2. Ordnung

$$(yzx)^2 - 4\,(yz)\,(zx)\,(xy) = 0 \tag{16}$$

d verlaufen ganz im grösseren Vierseit. So erhält man schliesslich eine Ein- ht in die Vertheilung der Gebiete für x, in denen

$$(y, z) - (y, x) - (x, z) \geqq 0$$

und der anderen, in denen

$$(y, z) - (y, x) - (x, z) \leqq 0.$$

: Die ersten werden in der Figur 7 durch verticale Schraffen, die zweiten, er vielmehr *das* zweite Gebiet, durch horizontale Schraffen bezeichnet.

Nunmehr ergiebt sich, ganz wie zuvor im Falle hyperbolischer Maassbe- mmung, der Begriff eines *Weges* zwischen y und z, der in dem kleineren erseit verläuft, und der Satz, dass unter allen diesen Wegen der gerade der gste ist. Wir behaupten aber weiter, dass in dem grösseren Vierseit dieselben ziehungen gelten. Das heisst, wir behaupten, dass in dem Dreieck y, x, z die nge der Seite yz nicht kleiner ist als die Summe der Längen der beiden deren Seiten, wenn x in dem grösseren Vierseit liegt, und wenn alle drei ngen durch das geradlinig erstreckte Integral über das Bogenelement so erklärt d, dass der Integrationsweg das grössere Vierseit nicht überschreitet.

Man erkennt ohne Schwierigkeit, dass man bei dem Beweise dieser Be- uptung drei Fälle zu unterscheiden hat.

1) Der Punkt x liegt in dem mittleren der oben unterschiedenen Theilgebiete des grösseren Vierseits (oder an dessen Grenze). Dann ist $(yz)\ (yx)\ (zx) \geqq 0$; die drei Längen sind $[y, z]$, (y, x) und (x, z); die Behauptung ist

$$[y, z] \geqq (y, x) + (x, z).$$

2) und 3). Der Punkt x liegt in einem der äusseren Theilgebiete — z. B. in dem, das von der Polare des Punktes y abgeschnitten wird, oder so, dass $(y, x) \geqq (x, z)$. Dann ist $(yz)\ (yx)\ (zx) \leqq 0$, die drei Längen sind $[y, z]$, $[y, x]$, (x, z), und die Behauptung ist

$$[y, z] \geqq [y, x] + (x, z), \text{ oder}$$
$$(y, x) \geqq (y, z) + (z, x).$$

Der Beweis ergiebt sich in allen Fällen durch die schon mehrfach angewendete Schlussweise.

Indem wir nun das erhaltene Resultat formuliren, werden wir uns kürzer fassen als zuvor, und im Wesentlichen nur hervorheben, was den gefundenen Satz vom vorigen unterscheidet.

Die verwendete Art der Maassbestimmung bezeichnen wir als *pseudo-elliptisch* deshalb, weil sie mit einer gewöhnlichen elliptischen im positiven Krümmungsmaasse (+ 1) übereinstimmt, sich aber von ihr durch das Vorhandensein reeller Wege von der Länge Null wesentlich unterscheidet. Unter Verwendung dieser Terminologie können wir dann sagen:

Sind y, z zwei verschiedene Punkte, deren gerade Verbindungslinie den absoluten Kegelschnitt nicht trifft, so gibt es zwei Familien von directen Wegen zwischen y und z.

Es giebt nämlich zwei von Minimalgeraden begrenzte einfache Vierseite, in denen y und z gegenüberliegende Ecken sind. Im Inneren oder an der Grenze je eines dieser Vierseite verlaufen die Wege der einen oder anderen Familie.

Unter allen Wegen derselben Familie ist bei pseudo-elliptischer Maassbestimmung der gerade der längste.

Jeder directe Weg ist ein Curvenstück von bestimmter Bogenlänge, dessen sämmtliche Sehnen den absoluten Kegelschnitt entweder gar nicht treffen oder höchstens berühren. Er wird durch irgend einen seiner Punkte x, der von y und z verschieden ist, in zwei Wege von gleicher Eigenschaft zerlegt. Wenn in dem zuvor ausgeschlossenen Grenzfall y und z ideale Punkte bleiben, aber die Gerade yz den absoluten Kegelschnitt berührt, so giebt es *einen* „eigentlichen" und *einen* „uneigentlichen" directen Weg zwischen y und z. Beide sind gerade.

Der erste verläuft ganz ausserhalb des absoluten Kegelschnittes; er hat die Länge Null. Die Länge des zweiten Weges ist nicht eigentlich definirt, darf aber $= \pi$ gesetzt werden.

Ein Curvenstück kann die Eigenschaft haben, dass sich um jeden seiner Punkte herum ein Gebiet abgrenzen lässt, in dem das Curvenstück directer Weg ist zwischen je zweien seiner Punkte y', z', ohne dass ihm im Ganzen nothwendig dieselbe Eigenschaft zukäme. Für solche Curvenstücke, die zwei gegebene Punkte y, z verbinden, gilt dann der letzte Satz nicht mehr. Im Gegentheil kann man offenbar zwei ganz beliebige ideale Punkte — also auch solche, deren gerade Verbindungslinie den absoluten Kegelschnitt in getrennten Punkten schneidet — durch solche Curvenstücke verbinden, deren Länge eine beliebig gegebene Zahl übertrifft. Die Länge des directen Weges zwischen Punkten y, z — falls es einen solchen Weg gibt — ist also in viel beschränkterem Sinne Maximum als die zuvor betrachtete Wegelänge bei hyperbolischer Maassbestimmung.

Da die Einführung der sogenannten idealen Punkte in die Geometrie *Lobatschewsky's* nicht gerade eine Nothwendigkeit ist, so wird vor der hier gewählten Ausdrucksform der aufgestellten Sätze unter Umständen den Vorzug verdienen eine andere, bei der jene Begriffsbildung vermieden wird. Natürlich lässt sich die Uebersetzung aus einer Sprache in die andere ohne Weiteres ausführen. Zum Beispiel lautet der Satz, der unserer ersten Reihe von Betrachtungen zu Grunde liegt, nunmehr wie folgt:

Wenn von drei Geraden y, x, z, die zu zweien entweder parallel oder ultraparallel sind, die mittlere die beiden anderen trennt, so ist die Länge der gemeinsamen Normale zwischen y und z grösser als die Summe der Längen der Normalen zwischen y, x und x, z, oder mindestens dieser Summe gleich.

Der Fall der Gleichheit tritt ein, wenn alle drei Geraden eine gemeinsame Senkrechte zulassen, oder, im Grenzfall, parallel sind. „Länge der gemeinsamen Normale zwischen y, z“ heisst natürlich, wenn y, z ultraparallel sind, die Länge der *kürzesten* Verbindungslinie zwischen einem Punkte von y und einem Punkte von z, und, im Grenzfalle des Parallelismus, die Null. Alles Weitere ergiebt sich von selbst: Der ideale Schnittpunkt von zwei ultraparallelen Geraden y, z oder seine Polare, die gemeinsame Senkrechte von y, z, betrachtet als Ort der zu ihr senkrechten Geraden, erscheint nun als *längster Weg* zwischen diesen beiden Geraden.

In ähnlicher Weise kann man unseren zweiten Satz mit seinen Folgerungen übertragen. Die grundlegende Thatsache erweist sich in diesem Falle als nicht verschieden von dem bekannten Satze über die Dreieckswinkel, der etwa so ausgesprochen werden kann:

„In jedem gewöhnlichen Dreiseit ist irgend ein Aussenwinkel grösser als die Summe der beiden Innenwinkel, die der dem Aussenwinkel gegenüberliegenden Seite anliegen, oder er ist mindestens dieser Summe gleich. Und zwar tritt Gleichheit dann und nur dann ein, wenn die drei Seiten durch einen Punkt gehen.“

Es ist minder anschaulich, mit Geraden und ihren Winkeln, als mit Punkten und ihren Entfernungen zu operiren; im ersten Falle hat man nicht wie im zweiten ein leitendes Princip in analogen Verhältnissen der Euklidischen Geometrie; und darin liegt jedenfalls der Grund dafür, dass man diesem längst bekannten Satze noch nicht die Wendung gegeben hat, auf die es uns hier ankommt (falls in der That die Sache neu sein sollte). Aber wenn wir hier die gerade Linie als — wenn auch nur sehr relatives — *Maximum* hinstellen, so handelt es sich gewiss um eine Thatsache, die in einem vollständigen System der Nicht-Euklidischen Geometrie nicht fehlen sollte. Ueberhaupt darf es, wie uns scheint, einmal gesagt werden, dass die Entwickelung der Nicht-Euklidischen Geometrie selbst in sehr elementaren Dingen noch ganz unvollständig ist. Die Lobatschewsky'sche Trigonometrie z. B. ist gewiss ein besonders wichtiges Bruchstück, aber immer nur ein Bruchstück: Dreiecke mit ganz oder theilweise ultraparallelen Seiten, insbesondere die den ersten entsprechenden Sechsecke mit lauter rechten Winkeln, müssen auch betrachtet werden in einem System, das einigen Anspruch auf Vollständigkeit erheben will. Auch dem Vorhandenen fehlt noch die feinere Durchbildung, das Studium der zugehörigen Gruppen, der Zusammenhang mit der Invariantentheorie u. s. w., obwohl die Hülfsmittel zu solchen Untersuchungen längst ausgebildet sind.*)

Fragen wir nunmehr nach einer etwanigen Ausdehnung der abgeleiteten Sätze auf eine beliebige Dimensionenzahl, so zeigt sich eine bemerkenswerthe Verschiedenheit der skizzirten kleinen Theorien.

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*) Die in der Litteratur vorhandenen Methoden und Formeln sind für solche Zwecke völlig ausreichend. S. *W. Jacobsthal*, Art. Trigonometrie in der Encyclopædie der elementaren Geometrie (Leipzig, 1905) und *E. Study* Sphärische Trigonometrie etc. (Abh. d. Sächs. G. D. W. Leipzig, 1893). Geometrie der Dynamen, Leipzig, 1903, S. 209–212.

Unser erster Satz über die Gerade als längsten Weg lässt sich auf eine unbestimmte Dimensionenzahl ausdehnen, der zweite ist auf die Ebene beschränkt.

Ist nämlich das absolute Polarsystem gegeben durch

$$(xy) = x_1y_1 - x_2y_2 - \ldots\ldots - x_ny_n = 0,$$

so können wir zunächst zwei ausserhalb des absoluten Punktgebildes $(xx) = 0$ gelegene Punkte y, z so annehmen, dass ihre gerade Verbindungslinie dieses Gebilde trifft. In jeder Ebene durch die Gerade yz erhalten wir dann die zuvor untersuchten Dreiecksfiguren. Es entsteht, an Stelle des — bei hyperbolischer Maassbestimmung — erörterten Vierseits ein $(n-1)$-dimensionales Raumstück, begrenzt durch Stücke zweier von y und z ausgehender Conoide,

$$(yx)^2 - (yy)\,(xx) = 0, \quad (zx)^2 - (zz)\,(xx) = 0.$$

„Wege" in diesem Raumstück sind im Wesentlichen so zu definiren, wie zuvor, nur brauchen sie nicht in Ebenen zu liegen. Der gerade Weg ist unter ihnen allen der längste.

Wenn dagegen die Gerade yz das absolute Punktgebilde nicht trifft, so giebt es drei verschiedene Arten von ebenen Wegen zwishen y, z. Eine Ebene durch y, z kann zunächst das absolute Gebilde in einem reellen Curvenzug durchdringen. Bei pseudo-elliptischer Maassbestimmung zeigt sich (y, z) oder $[y, z]$ als Maximum zunächst unter geeigneten gebrochen-geradlinigen Wegen. Sodann kann die Ebene das absolute Gebilde berühren. Man erhält in diesem Falle, wie sich unschwer einsehen lässt, unendlich viele gleichlange Wege zwischen beiden Punkten. Endlich kann die Ebene das absolute Gebilde gar nicht treffen. Dann wird (y, z) ein Minimum. *Es ergiebt sich also, dass die Länge des geraden Weges in diesem Falle überhaupt nicht Extremum ist.*

Das schliesst natürlich nicht aus, dass auf einer Mannigfaltigkeit niederer Dimension, die einer solchen Maassbestimmung unterworfen wird, das Problem der kürzesten (oder längsten) Linien doch wieder einen klaren Inhalt bekommen kann. So giebt es im Raume von acht Dimensionen eine Mannigfaltigkeit M_4 von vier Dimensionen, der bei passend gewählter pseudo-elliptischer Maassbestimmung die folgende merkwürdige Eigenschaft zukommt:

*Die Länge des auf M_4 selbst gemessenen kürzesten Weges zwischen zwei Punkten y, z von M_4 ist gleich der Länge des geraden Weges zwischen eben diesen Punkten.**

Andere Erweiterungen des Vorgetragenen wollen wir nur andeuten.

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*) Math. Ann. Bd. 60, 1905, S. 361.

Aehnliche Extremums-Eigenschaften wie die hier betrachteten werden sich finden bei den geodätischen Linien, die zu einem Bogenelement gehören, dessen Quadrat durch eine quadratische Differentialform erklärt ist, in dem Falle, wo diese Form für die Linienelemente eine hyperbolische Maasbestimmung definirt. Ferner kann man bei einem Theile der vorausgehenden Betrachtungen den Kegelschnitt durch irgend ein Oval ersetzen. Es gehört ferner in diesen Gedankenkreis z. B. der folgende unschwer zu begründende aber doch vielleicht nicht ganz uninteressante Satz:

Es möge eine im endlichen Gebiet der Euklidischen Ebene gelegene Menge von Punkten erstens ein abgeschlossenes Continuum bilden und zweitens die Eigenschaft haben, dass alle Sehnen zwischen verschiedenen Punkten der Menge mit einer bestimmten Geraden Winkel ϕ bilden, die einer Ungleichung der Form

$$|tg\phi| \leq M$$

genügen. Dann ist diese Punktmenge ein rectificirbares Curvenstück, das gewisse zwei Punkte y, z mit einander verbindet.

Ist D der Abstand der Projectionen der Punkte y, z auf die genannte Gerade, so ist die Bogenlänge des Curvenstücks höchstens gleich

$$|D\sqrt{1+M^2}|.$$

II.

Die Begriffe Links und Rechts in der elliptischen Geometrie.

In allen Untersuchungen über Nicht-Euklidische Geometrie, die mit der tiefer liegenden Problemen allein gewachsenen analytischen Methode geführt werden, trifft man gleich bei den ersten Schritten auf gewisse Quadratwurzeln*), deren Vorzeichen eine geometrische Bedeutung haben. Diese Deutung, die im Gebrauche solcher Worte wie Richtung, Drehungssinn, links und rechts zum Ausdruck kommt, wird natürlich erst möglich auf Grund besonderer Definitionen. Bei diesen hat die Willkür einen gewissen Spielraum — es handelt sich jedesmal um eine Entscheidung zwischen zwei logisch-gleichwerthigen Möglichkeiten. Man kann die nöthigen Bestimmungen so zu treffen suchen, dass bei geeignetem Grenzübergang zum Euklidischen Raume die für diesen bereits üblichen Erklärungen entstehen.

Man sollte meinen, dass Begriffen von so tief einschneidender Bedeutung von vorn herein die grösste Aufmerksamkeit geschenkt worden wäre, dass man für sie sorgfältig abgewogene jede Mehrdeutigkeit ausschliessende Erklärungen gegeben haben müsste. Soviel dem Verfasser bekannt, hat man sich jedoch um diese Wurzelvorzeichen in der Regel überhaupt nicht gekümmert, oder man hat Worte so gebraucht, als ob ihnen von selbst ein klarer Sinn zukommen könnte. Dass unter diesen Umständen auch der analytische Apparat eines sorgfältigeren Studiums und weiterer Ausbildung bedarf, wird einleuchtend sein. — In emd vorliegenden Beitrag betrachten wir insbesondere den dreidimensionalen sogenannten elliptischen Raum. Hier (oder auch im sphärischen Raum) findet die Deutung der erwähnten Wurzelgrössen eine besonders merkwürdige Anwendung in einem Uebertragungsprincip, das die Nicht-Euklidische mit der Euklidischen Geometrie in Zusammenhang bringt.

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*) Siehe etwa *Lindemann*, Math. Ann. Bd. VII (1874), S. 56–144.

Im Mittelpunkte unserer Darlegung stehen zwei nahe verwandte Begriffe, die im reellen Gebiete mit einander zusammenfallen, und als *Speer* und *Pfeil* bezeichnet werden. Ein *reeller* Speer oder Pfeil ist eine sogenannte orientirte Gerade, oder Gerade mit positiver Richtung.*) Wir werden unter Anderem zeigen:

Bei geeigneter Wahl des Raumelementes (nämlich des Speeres oder Pfeiles, nicht aber der geraden Linie) wird die reelle sogenannte elliptische Geometrie völlig identisch mit der Elementargeometrie der Punktepaare, die man zwei gleichgrossen Kugeln entnehmen kann.†)

Damit soll natürlich gesagt werden, dass diese beiden Arten der Geometrie in logischer Hinsicht nicht unterschieden zu werden brauchen, dass sie, vom Standpunkte des Logikers betrachtet, ein einziges System bilden. Aber wir behaupten auch — was uns ungleich wichtiger zu sein scheint — dass dieses System von Begriffen und Lehrsätzen sich auf sehr verschiedene Arten, und zwar an bemerkenswerthen Stellen, in das Ganze der Geometrie einfügt.

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*) In der Euklidischen Geometrie hat man derartige Figuren betrachtet, seit man negative Grössen zu gebrauchen gelernt hat. Von *Mölius*, *Laguerre*, *Stephanos*, *Lie* und Anderen sind sie vielfach verwerthet worden.

Die Methode des Verfassers unterscheidet sich von denen seiner Vorgänger unter Anderem durch die Darstellung dieser Figuren durch besondere und zwar homogene *Coordinaten*.

†) In summarischer Darstellung ist dieses Uebertragungsprincip vom Verfasser bereits angegeben und durch Anwendungen erläutert worden (Festschrift der Universität Greifswald, 1900; mit einem Zusatz abgedruckt im Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 11, 1902, S. 340 u. ff.). Aehnliche Gedanken, allerdings mit nicht ganz genügender Ausarbeitung der Begriffe, hat gleichzeitig auch Herr *Joh. Petersen* entwickelt, dessen Arbeit (Kopenhagener Akademieber. 1900, S. 306 u. s. f.) Freunde der Nicht-Euklidischen Geometrie mit Interesse lesen werden. (Herr Petersen hat seitdem seinen Namen in *Hjelmslev* umgeändert.)

Weitere Anwendungen findet man in Schriften von *J. Coolidge* (The Dual Projective Geometry of Elliptic and Spherical Space, Diss. Bonn, 1904; Les Congruences isotropes, Acc. R. die Torino, 1903, 1905), sowie in einer Arbeit des Verfassers, Jahresber. d. D. M. V., Bd. 15, 1906, S. 476, u. f.

Verschieden ist dem entsprechend in beiden Fällen sowohl die Terminologie (z. B. Speer $\equiv$ Punktepaar) als auch der zugehörige Vorstellungsinhalt; und was im einen Falle anschaulich und naheliegend ist, braucht es nicht ebenso auch im anderen zu sein. Das aber, kann man sagen, sind die Bedingungen, unter denen ein solches Uebertragungsprincip, deren man ja schon mehrere kennt, sich als wirklich fruchtbringend erweisen wird.

Uebrigens ist der bezeichnete Stoff ungemein umfangreich. Wir werden deshalb in dieser einleitenden Arbeit nur die grundlegenden Thatsachen vorführen können, und auch diese — in der Hauptsache — nur in der Beschränkung auf *reelle* Figuren. Sogenannte imaginäre Figuren ziehen wir soweit in Betracht, als sie zum besseren Verständniss der reellen Geometrie dienen. Alle einzuführenden Grössen sind reell, wofern nicht ausdrücklich das Gegentheil bemerkt wird.

Die Geraden im dreidimensionalen elliptischen Raume bilden das gewöhnliche Plücker'sche Liniencontinuum. Wir werden nun neben dem Begriff der Geraden vorläufig noch zwei andere Begriffe verwenden, die wir als *Speer* und *Linienkreuz* (oder, in anderem Zusammenhang, Strahlenkreuz) bezeichnen. Das Linienkreuz ist die Zusammenfassung einer Geraden mit ihrer absoluten Polare. Diese treten nämlich häufig verbunden auf, und man braucht daher für ihre Zusammenstellung ein besonderes Wort.* Der Name ist darin begründet, dass die Figur, von irgend einem nicht zu ihr gehörigen Punkte aus betrachtet, als Kreuz mit rechtem Winkel erscheint. „Speer" ist, wie gesagt, eine kürzere Bezeichnung für „orientirte Gerade," oder Gerade „mit positiver Richtung", besser — da die Gerade geschlossen ist — Gerade mit positivem *Umlaufssinn.* Sind zwei Punkte x, y auf einer *Geraden* gegeben, und kennt man, wie wir annehmen wollen, den Cosinus ihrer Entfernung, so kann daraus deren Sinus durch Ausziehen einer Quadratwurzel, also zweiwerthig, bestimmt werden. Bei zwei Punkten auf einem *Speer* sind dagegen Cosinus und Sinus der Entfernung zugleich rational-bekannt. Die Mehrdeutigkeit der Entfernung ist, so weit sie das Vorzeichen des Sinus beeinflusst, aufgehoben.

Jede Gerade gehört also einem Linienkreuz an, und sie wird von zwei Speeren überdeckt. Jedes Linienkreuz enthält zwei Gerade, die einander im absoluten Polarsystem entsprechen, und es enthält also vier Speere. Daher bilden die drei Figuren Speer, Gerade, Linienkreuz im Sinne des Analysis Situs

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*) Es ist diese Figur das einfachste Raumelement einer besonderen Disciplin, der dual-projectiven Geometrie im elliptischen Raume. Vgl. die citirten Arbeiten von *Coolidge* und dem Verfasser.

verschiedene Mannigfaltigkeiten. Nur in einer gewissen Umgebung eines bestimmten Speeres kann man sie eindeutig auf einander beziehen. Wo immer aber der ganze Raum betrachtet wird, also bei allen algebraischen Problemen, da ist es unerlässlich, diese Figuren gehörig zu unterscheiden.

Die analytische Erklärung der dreierlei Gebilde gestaltet sich am einfachsten, wenn man das absolute Polarsystem in der Form annimmt:

$$(xy) = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0. \tag{1}$$

Die gewöhnlichen durch die Gleichung

$$\mathfrak{X}_{01}\mathfrak{X}_{23} + \mathfrak{X}_{02}\mathfrak{X}_{31} + \mathfrak{X}_{03}\mathfrak{X}_{12} = 0 \tag{2}$$

verbundenen Liniencoordinaten $\mathfrak{X}_{i\kappa} = x_iy_\kappa - x_\kappa y_i$ können dann durch andere Verhältnissgrössen $X_1 \ldots X_6$ vertreten werden, die wie folgt erklärt sind:

$$\begin{aligned} 2X_1 &= \mathfrak{X}_{01} + \mathfrak{X}_{23}, & 2X_2 &= \mathfrak{X}_{01} - \mathfrak{X}_{23}, \\ 2X_3 &= \mathfrak{X}_{02} + \mathfrak{X}_{31}, & 2X_4 &= \mathfrak{X}_{02} - \mathfrak{X}_{31}, \\ 2X_5 &= \mathfrak{X}_{03} + \mathfrak{X}_{12}, & 2X_6 &= \mathfrak{X}_{03} - \mathfrak{X}_{12}, \end{aligned} \tag{3}$$

so dass die Gleichung (2) die Form

$$X_1^2 + X_3^2 + X_5^2 = X_2^2 + X_4^2 + X_6^2 \tag{4}$$

annimmt.

Als Coordinaten einer Geraden können also sechs Verhältnissgrössen $X_1 : X_3 : X_5 : X_2 : X_4 : X_6$ dienen, die durch die Gleichung (4) *verbunden sind.*

Nun findet sich sogleich, dass der Uebergang von einer Geraden zu ihrer absoluten Polare dadurch bewirkt wird, dass man entweder X_2, X_4, X_6 oder X_1, X_3, X_5 durch die entgegengesetzt-gleichen Grössen ersetzt. Sind anderseits zweimal drei (reelle) Verhältnissgrössen $\mathfrak{x}_1$, $\mathfrak{x}_3$, $\mathfrak{x}_5$ und $\mathfrak{x}_2$, $\mathfrak{x}_4$, $\mathfrak{x}_6$ gegeben, so kann man immer auf *zwei* wesentlich verschiedene Arten zugehörige Grössen

$$\begin{aligned} X_1 &= \kappa\,\mathfrak{x}_1, & X_3 &= \kappa\,\mathfrak{x}_3, & X_5 &= \kappa\,\mathfrak{x}_5, \\ X_2 &= \lambda\,\mathfrak{x}_2, & X_4 &= \lambda\,\mathfrak{x}_4, & X_6 &= \lambda\,\mathfrak{x}_6 \end{aligned}$$

derart finden, dass die Gleichung (4) besteht, und diese beiden durch Ausziehen einer Quadratwurzel zu ermittelnden Systeme von sechs homogenen Grössen stellen absolut-polare gerade Linien dar. Also ergiebt sich:

Als Coordinaten eines Linienkreuzes können zwei Systeme von je drei Verhältnissgrössen

$$\mathfrak{x}_1 : \mathfrak{x}_3 : \mathfrak{x}_5 \text{ und } \mathfrak{x}_2 : \mathfrak{x}_4 : \mathfrak{x}_6$$

dienen.

Endlich behaupten wir, dass Coordinaten eines Speeres entstehen, wenn man aus den beiden unter einander gleichen Ausdrücken (4) die Quadratwurzel zieht und diese den Grössen X als eine siebente Coordinate $X_0 = \frac{1}{2}\mathfrak{X}_{00}$ hinzufügt:

Als Coordinaten eines Speeres können sieben Verhältnissgrössen dienen,

$$X_0 : X_1 : X_3 : X_5 : X_2 : X_4 : X_6,$$

die durch die beiden Gleichungen

$$X_0^2 = X_1^2 + X_3^2 + X_5^2 = X_2^2 + X_4^2 + X_6^2 \tag{5}$$

verbunden sind.

Wo es sich, wie hier, um reelle Grössen handelt,*) da ist das Verschwinden von X_0 ausgeschlossen, und man kann daher, *im reellen Gebiete*, unbeschadet der Vollständigkeit der analytischen Darstellung, $X_0 = 1$ setzen, und als Coordinaten eines Speeres sechs nicht-homogene Grössen benutzen, die durch die beiden Gleichungen

$$X_1^2 + X_3^2 + X_5^2 = 1\,, \quad X_2^2 + X_4^2 + X_6^2 = 1 \tag{6}$$

verbunden sind.

In der That ergiebt sich das Gesagte sogleich, wenn man — unter Annahme des Krümmungsmaasses *Eins* für die elliptische Maassbestimmung — Sinus und Cosinus der Entfernung zweier Punkte x, y analytisch ausdrückt.

Man hat zunächst, wie bekannt,

$$\cos\,(x, y) = \frac{(xy)}{\sqrt{(xx)}\,\sqrt{(yy)}}, \tag{7}$$

und folglich, wenn x und y verschieden sind, und $\mathfrak{X}$ oder X die dann bestimmte Verbindungslinie beider Punkte bezeichnet,

$$\sin\,(x, y) = \frac{\mathfrak{X}_{00}}{\sqrt{(xx)}\,\sqrt{(yy)}} \cdot \frac{x_i y_\varkappa - x_\varkappa y_i}{\mathfrak{X}i\varkappa}. \tag{8}$$

Die Grösse $\sin\,(x, y)$ enthält also gegenüber $\cos\,(x, y)$ keine neue Irrationalität, und $\operatorname{tg}\,(x, y)$ ist von solchen überhaupt frei.

Natürlich ist es nicht nothwendig, den letzten Satz gerade so zu fassen, wie wir es gethan haben. Die Grössen $\mathfrak{X}_{00}$, $\mathfrak{X}_{i\varkappa}$ z. B., die durch die Gleichungen (2) und

$$\mathfrak{X}_{00}^2 = \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2 + \mathfrak{X}_{23}^2 + \mathfrak{X}_{31}^2 + \mathfrak{X}_{12}^2 \tag{9}$$

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*) Im complexen Gebiete wird man umgekehrt die Coordinaten zur *Definition* der erweiterten Begriffe Gerade, Linienkreuz, Speer, benutzen. Diese complexen Figuren haben dann aber nicht mehr einen so einfachen Zusammenhang wie die ebenso benannten reellen, auch sind noch ganz anders geartete Coordinaten in Betracht zu ziehen. Wir kommen hierauf später noch zurück.

verbunden sind, leisten Dasselbe. Von zwei Speeren, deren Coordinaten entgegengesetzte Verhältnisse $X_0 : X_\iota$ oder $\mathfrak{X}_{00} : \mathfrak{X}_{\iota\kappa}$ liefern, werden wir sagen, jeder sei die *Umkehrung* des anderen.

Dieser Umkehrungsprocess ist mit allen Bewegungen im elliptischen Raume vertauschbar, selbst aber ist er keine Bewegung: durch ihn wird die Gruppe der Bewegungen, mit dem Speer als Raumelement, zu einer sogenannten gemischten Gruppe *erweitert.**)

Unsere Formeln ergeben nun ohne Weiteres die folgenden Sätze:

Das Continuum aller reellen Speere im elliptischen (oder sphärischen) Raume lässt sich, überall eindeutig und stetig, abbilden auf das Continuum aller reellen Paare von Punkten, die man zwei Kugeln (des reellen Euklidischen Raumes) vom Radius Eins entnehmen kann.)†

Bei sachgemässer Wahl dieser Abbildung gehören je zwei Punktepaare, die auf beiden Kugeln einander diametral gegenüberliegen, zur selben GERADEN; *und je vier Punktepaare, die auf denselben beiden Durchmessern der Bildkugeln liegen, zum selben* LINIENKREUZ.

Mit anderen Worten: Das Continuum der Linienkreuze ist abgebildet auf das Continuum der Geradenpaare aus zwei Bündeln, oder der Punktepaare aus zwei ebenen projectiven Punktcontinuis. Durch passend gewählte vierfache Ueberdeckung dieses Continuums entsteht ein weiteres Continuum, auf das das Speercontinuum abgebildet werden kann. Die Punktepaare aus dem zweiten Continuum sind durch den Umstand, dass dieses das erste überlagert, zu Paaren angeordnet; und wenn man je zwei solche auf beide Kugeln vertheilte Paare nicht unterscheidet, so entsteht ein drittes Continuum vom Zusammenhang des *Plücker*'schen Liniencontinuums. Alle drei Continua sind abgeschlossen.

Natürlich liegt die Bedeutung des abgeleiteten Satzes nur zum kleinsten Theile im Gebiete der Analysis situs. Es kommt vielmehr vor Allem darauf an, dass der vorgeführten besonderen Form der Abbildung der Charakter der *Invarianz* gegenüber Bewegungen des elliptischen Raumes sowohl wie Bewegungen der beiden Kugeln innewohnt. Dies wollen wir nunmehr darlegen. Wir bezeichnen zu diesem Zwecke die beiden Kugeln, deren reelle Punkte die homogenen rechtwinkligen Coordinaten $X_0 : X_1 : X_3 : X_5$ und $X_0 : X_2 : X_4 : X_6$ haben, als *linke und rechte Bildkugel.* Wir brauchen ferner die Bezeichnungen und Rechnungsregeln der Quaternionentheorie. Wir verstehen zum Beispiel unter $\dot{x}$, $\dot{X}_l$, $\dot{X}_r$, $\dot{A} = \dot{a}$, $\dot{B} = \dot{b}$ die Quaternionen

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*) Der Speer verhält sich zur Geraden etwa so wie in der *zweidimensionalen* sphärischen Geometrie der Punkt der Kugel zu deren Durchmesser, oder zum Punkte der elliptischen Ebene. Bewegungen entsprechen in dieser Analogie Bewegungen, der Umkehrung aber entspricht die Diametralspiegelung. Der erweiterten Bewegungsgruppe ist analog die Gruppe der Bewegungen und Umlegungen.

†) Aus dem Folgenden ergiebt sich, dass man statt dieser Kugeln auch Kugeln oder besser noch doppelt überdeckte Ebenen des elliptischen Raumes selbst benutzen kann.

$$\begin{aligned}
\dot{x} &= x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3,\\
\dot{X}_l &= X_0 e_0 + X_1 e_1 + X_3 e_2 + X_5 e_3,\\
\dot{X}_r &= X_0 e_0 + X_2 e_2 + X_4 e_2 + X_6 e_3,\\
\dot{A} &= a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 = \dot{a},\\
\dot{B} &= b_0 e_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 = \dot{b},
\end{aligned}$$

u. s. w., während x, X_l, X_r selbst die den drei ersten zugeordneten Punkte des elliptischen Raumes und der beiden Bildkugeln bezeichnen. Die Conjugirte $a_0 e_0 - a_1 e_1 - a_2 e_2 - a_3 e_3$ der Quaternion $\dot{A}$ bezeichnen wir mit $\bar{\dot{A}}$. Jede Bewegung im elliptischen Raume kann dann bekanntlich auf eine einzige Weise in eine *linkseitige Schiebung*

$$\dot{x}' = \bar{\dot{a}} . \dot{x} \qquad \{N(\dot{a}) = \dot{a} . \bar{\dot{a}}, \neq 0\} \tag{10,l}$$

und eine *rechtseitige Schiebung*

$$\dot{x}' = \dot{x} . \dot{b} \qquad \{N(\dot{b}) = \dot{b} . \bar{\dot{b}}, \neq 0\} \tag{10,r}$$

zerlegt, und folglich in der Form

$$\dot{x}' = \bar{\dot{a}} . \dot{x} . \dot{b} \qquad \{N(\dot{a}),\ N(\dot{b}) \neq 0\} \tag{11}$$

dargestellt werden. Die Zusammensetzung der Parameter mehrerer hinter einander auszuführender Schiebungen und überhaupt mehrerer Bewegungen erfolgt nach der Multiplicationsregel der Quaternionen

$$\dot{a} . \dot{a}' = \dot{a}'', \ \dot{b} . \dot{b}' = \dot{b}''.\text{*} \tag{12}$$

Das heisst, führt man z. B. *nach* der Schiebung mit den Parametern a_i die

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*) Die obigen Formeln unterscheiden sich von den bekannten *Cayley's* (Crelle's Journal, Bd. 50, oder Werke Bd. II, S. 214) nur ganz nebensächlich in der Bezeichnung. Die geringe Abweichung, die wir angebracht haben, indem wir in (11) *links* nicht die Quaternion $\dot{a}$, sondern *deren Conjugirte* setzen, bewirkt aber eine nicht unbedeutende Vereinfachung, wo es sich darum handelt, Folgerungen aus diesen Formeln durch Worte wiederzugeben; sonst müsste schon der Inhalt der Gleichungen (12) auf eine etwas umständlichere Weise beschrieben werden. Es ist überhaupt zu beachten, dass und wie die genaue Form der später abzuleitenden Sätze von der Auswahl abhängt, die an mehreren Stellen willkürlich, aber mit nothwendiger Willkür, unter logisch-gleichwerthigen Bestimmungen getroffen wird.

Schiebung mit den Parametern a'_i aus, so ergiebt sich die Schiebung mit den Parametern a''_i.

Ferner ist evident die Folgerung: *Die linkseitigen und rechtseitigen Schiebungen, und damit weiterhin alle durch die Bezeichnungen links und rechts zu unterscheidenden Begriffe werden durch die Umlegungen mit einander vertauscht.* (Unter einer Umlegung verstehen wir hier wie sonst eine Transformation mit spiegelbildlicher Gleichheit entsprechender Figuren).

Nimmt man nun an, dass $N(\dot{a}) = N(\dot{b})$ — was unbeschadet der Allgemeinheit auf zwei Arten bewirkt werden kann — und unterwirft man dann den Speer X der Bewegung (11), so erhält man die einfachen Formeln

$$\dot{X}_l' = \dot{\bar{A}}.\dot{X}_l\dot{A}, \quad \dot{X}_r' = \dot{\bar{B}}.\dot{X}_r.\dot{B}, \tag{13}$$

die folgenden Satz enthalten:

Den linkseitigen und rechtseitigen Schiebungen im elliptischen Raume entsprechen eindeutig-umkehrbar die Drehungen der linken und rechten Bildkugel.

Die (Cayley'schen) Parameter der Schiebungen sind identisch mit den (Euler'schen) Parametern der beiden orthogonalen Substitutionen, die die zugehörigen Drehungen der Bildkugeln bewirken.

Hiernach sind also die in Betracht kommenden Gruppen — die Gruppe der Bewegungen im elliptischen Raume, und die Gruppe der simultan auszuführenden Drehungen beider Kugeln, (*holoedrisch-*)*isomorph* auf einander bezogen.

Jeder Punkt im elliptischen Raume, und ebenso jede Ebene, ist nun Ort von ∞^2 Speeren. Die nächste Frage, die sich darbietet, wird also diese sein: Welches sind die sphärischen Bilder der ∞^3 Punkte und der ∞^3 Ebenen? Die Antwort hierauf muss abhängen von der Orientirung der beiden Cartesischen Coordinatensysteme $X_0 : X_1 : X_3 : X_5$ und $X_0 : X_2 : X_4 : X_6$. Wir nehmen an, dass diese *gleichartig orientiert* (congruent, z. B. identisch) sind. Dann gilt der Satz:

Den ∞^2 Speeren eines Bündels entsprechen solche ∞^2 Punktepaare beider Kugeln, deren einzelne Punkte durch eine Bewegung einander zugeordnet sind.	*Den ∞^2 Speeren eines Feldes entsprechen solche ∞^2 Punktepaare beider Kugeln, deren einzelne Punkte durch eine Umlegung einander zugeordnet sind.*
Der Scheitel des Bündels hat zu homogenen Coordinaten die Euler'schen Parameter dieser Bewegung.	*Die Ebene des Feldes hat zu homogenen Coordinaten die Euler'schen Parameter dieser Umlegung.*

Zunächst erkennt man nämlich ohne Weiteres, dass der folgende speciellere Satz gilt:

Damit zwei Speere X, Y einander schneiden, ist nothwendig und hinreichend, dass ihre sphärischen Bilder X_l, Y_l und X_r, Y_r auf beiden Kugeln zu einander congruent oder, was dasselbe ist, symmetrisch (symmetrisch-gleich) sind.

Es müssen also, wenn man die linke Kugel z. B. congruent — durch eine Bewegung — auf die rechte abbildet, dadurch aus den ∞^4 Speeren solche ∞^2 herausgehoben werden, die einander paarweise schneiden. Man braucht daher nur noch festzustellen, ob diese, oder vielmehr ihre geradlinigen Träger, ein Bündel (x) oder ein Feld (u) erfüllen.

Dass nun ein Punkt x und eine Gerade oder ein Speer X vereinigt liegen, wird ausgedrückt durch ein ohne weiteres aufzustellendes System von vier linearen Gleichungen (deren zwei von einander unabhängig sind). Diese Gleichungen lassen sich mit Hülfe der Quaternionen der Form nach in eine einzige zusammenziehen:

Der Punkt x und die Gerade oder der Speer X liegen vereinigt, wenn die entsprechenden Quaternionen $\dot{x}$ und $\dot{X}$ in der Beziehung stehen:

$$\dot{X}_l \,.\, \dot{x} = \dot{x} \,.\, \dot{X}_r; \tag{14a}$$

ebenso findet die vereinigte Lage einer Ebene u und eines Speeres oder einer Geraden X ihren Ausdruck in der Quaternionengleichung:

$$\dot{X}_l \,.\, \dot{u} = \dot{u} \,.\, \dot{\bar{X}}_r. \tag{14b}$$

Löst man aber diese Gleichungen nach der Quaternion $\dot{X}_r$ auf, so ergiebt sich

$$N(\dot{x}) \,.\, \dot{X}_r = \dot{\bar{x}} \,.\, \dot{X}_l \,.\, \dot{x}, \tag{15a}$$

$$N(\dot{u}) \,.\, \dot{X}_r = \dot{\bar{u}} \,.\, \dot{\bar{X}}_l \,.\, \dot{u}. \tag{15b}$$

Nun bedeuten die Gleichungen (15a) eine *eigentliche*, die Gleichungen (15b) eine *uneigentliche orthogonale Substitution*. Die Punkte X_l, d. h. $X_0 : X_1 : X_3 : X_5$ der linken Bildkugel werden also den Punkten X_r, d. h. $X_0 : X_2 : X_4 : X_6$ der rechten Bildkugel im Falle (a) durch eine Bewegung, im Falle (b) durch eine Umlegung zugeordnet. Anders ausgedrückt, im Falle (a) sind entsprechende Figuren auf beiden Kugeln zu einander congruent, im Falle (b) symmetrisch-gleich.

Die abgeleiteten Sätze sind von Bedeutung für die algebraische und Differentialgeometrie im elliptischen Raume, namentlich aber enthalten sie *in nuce* eine bestimmte Reihenfolge gesetzt sind). Ein entsprechender Satz gilt natürlich für Speere, die den vorgelegten Speer im selben Punkte senkrecht schneiden. Treffen solche Speere Y, Z die absolute Polare des Speeres X oder $\mathfrak{X}$ in den Punkten η, ζ, so ist die goniometrische Tangente des Winkels zwischen Y, Z, d. h. des Abstandes (η, ζ) nach Nr. (7, 8) gegeben durch die Formeln*)

$$tg\,(Y, Z) = \frac{\mathfrak{X}_{00}}{(\eta\,\zeta)} \cdot \frac{\eta_0\zeta_1 - \eta_1\zeta_0}{\mathfrak{X}_{23}} = * = * = \frac{\mathfrak{X}_{00}}{(\eta\,\zeta)} \cdot \frac{\eta_2\zeta_3 - \eta_3\zeta_2}{\mathfrak{X}_{01}} = * = * =$$
$$= \frac{2X_0}{(\eta\,\zeta)} \cdot \frac{\eta_0\zeta_1 - \eta_1\zeta_0}{X_1 - X_2} = * = * = \frac{2X_0}{(\eta\,\zeta)} \cdot \frac{\eta_2\zeta_3 - \eta_3\zeta_2}{X_1 + X_2} = * = *. —$$

Von unseren beiden Bildkugeln hatten wir bisher nur angenommen, dass die zugehörigen Cartesischen Coordinatensysteme gleich-orientirt seien. Darin liegt, dass eine sogenannte *Orientirung* der einen Kugel, d. h. die Festsetzung eines positiven Umlaufssinnes um deren Punkte herum, eine entsprechende Orientirung der anderen Kugel nach sich zieht. Danach haben wir noch zwei Möglichkeiten vor uns, und unter diesen müssen wir wiederum eine Auswahl treffen.

Es mögen die Punkte Y_l, Z_l und Y_r, Z_r von den Punkten X_l, X_r den sphärischen Abstand $\equiv \frac{1}{2}\pi$ mod. π haben. Dann kann man offenbar die Grössen $tg\,(Y_l, Z_l)$ und $tg\,(Y_r, Z_r)$ eindeutig erklären durch die Formeln

$$tg\,(Y_l, Z_l) = \frac{X_0(Y_3Z_5 - Y_5Z_3)}{(Y_1Z_1 + Y_3Z_3 + Y_5Z_5)\,X_1} = \text{etc.},$$
$$tg\,(Y_r, Z_r) = \frac{X_0(Y_4Z_6 - Y_6Z_4)}{(Y_2Z_2 + Y_4Z_4 + Y_6Z_6)\,X_2} = \text{etc.},$$

aber ebensowohl auch durch die entgegengesetzt-gleichen Ausdrücke; und es ist deutlich, dass die Entscheidung für die angeschriebenen Formeln eben darauf hinausläuft, um alle Punkte beider Kugeln herum einen positiven Drehungssinn festzusetzen. Um eine bestimmte Vorstellung zu erhalten, wollen wir annehmen, dass z. B. die linke Bildkugel die Erdkugel sei, und dass der Punkt (1, 1, 0, 0) deren Nordpol vorstelle. Lassen wir dann den Punkt (1, 0, 1, 0) in den Punkt (1, 0, 0, 1) vermöge einer Drehung durch 45° von Westen nach Osten übergehen, so ist damit der Aequator orientirt, und mit ihm die ganze Kugel. Irgend ein Punkt X_l liegt zu dem entsprechenden orientirten Hauptkreis so, wie der Nordpol zu dem von Westen nach Osten umlaufenen Aequator, also *links* von der positiven Richtung des Hauptkreises. Sind Y_l und $Y_l + d\,Y_l$ consecutive Punkte auf dem Hauptkreis, so folgen diese im positiven Sinne auf einander, wenn $(Y_l,\ Y_l + d\,Y_l) > 0$ ist.

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*) Wir benutzen das Zeichen (Y, Z) für diesen Winkel nur vorläufig. Später werden wir ein anderes Zeichen einführen.

Nun ist leicht zu sehen, dass zwischen den dreierlei Grössen, deren Ausdrücke wir soeben gebildet haben, eine einfache Beziehung immer dann besteht wenn die Punkte X_l, X_r sowie Y_l, Y_r und Z_l, Z_r die Bilder der Speere X und Y, Z sind. Die drei Ausdrücke müssen einander gleich sein, höchstens abgesehen von ihren Vorzeichen. Wir behaupten, dass ihre Gleichheit sich auch auf die Vorzeichen erstreckt, $tg\,(Y, Z) = tg\,(Y_l, Z_l) = tg\,(Z_r, Y_r)$.

Um dies einzusehen, braucht man nur zu zeigen, dass die vollkommene Gleichheit der drei Ausdrücke bei irgend einer durch Bewegungen zu erreichenden speciellen Lage der Speere X, Y, Z vorhanden ist: Dann muss sie offenbar immer stattfinden. Wir wollen etwa annehmen, dass der Punkt in dem der Speer X von den beiden Speeren Y, Z senkrecht getroffen wird, der Punkt $(1:0:0:0)$ sei. Dann wird

$$X_1 = X_2 = \mathfrak{X}_{01}, \qquad X_3 = X_4 = \mathfrak{X}_{02}, \qquad X_5 = X_6 = \mathfrak{X}_{03},$$

und $\mathfrak{X}_{23} = \mathfrak{X}_{31} = \mathfrak{X}_{12} = 0$, u. s. w., ferner $\eta_0 = \zeta_0 = 0$, und $z.\,B.$

$$Y_1 = Y_2 = \rho\eta_1, \qquad Y_3 = Y_4 = \rho\eta_2, \qquad Y_5 = Y_6 = \rho\eta\ ;$$

alle drei Ausdrücke reduciren sich jetzt auf denselben Werth, nämlich

$$\frac{X_0\,(\eta_2\zeta_3 - \eta_3\zeta_2)}{(\eta_1\zeta_1 + \eta_2\zeta_2 + \eta_3\zeta_3)\,\mathfrak{X}_{01}} = \text{etc.}$$

Der hiermit bewiesene Satz lässt sich am bequemsten in kinematischer Form ausdrücken:

Lässt man um einen Speer X einen zweiten Y, der den ersten senkrecht trifft, im positiven Sinne sich mit constanter Winkelgeschwindigkeit drehen, so drehen sich beide sphärischen Bilder von Y um die sphärischen Bilder von X ebenfalls im positiven Sinne, und zwar mit eben derselben Winkelgeschwindigkeit.

Die abgeleiteten Formeln aber lassen sich offenbar zweckmässiger schreiben. Wir setzen zur Abkürzung

$$\begin{aligned}(XYZ)_l &= |X_1 Y_3 Z_5|,\\ (XYZ)_r &= |X_2 Y_4 Z_6|,\\ (YZ)_l &= Y_1 Z_1 + Y_3 Z_3 + Y_5 Z_5,\\ (YZ)_r &= Y_2 Z_2 + Y_4 Z_4 + Y_6 Z_6.\end{aligned}$$

zeigt, dass es zweckmässig ist, „*linkseitig*" die Erzeugenden zu nennen, die von den linkseitigen Schiebungen unter einander vertauscht werden und bei den rechtseitigen Schiebungen einzeln in Reihe bleiben. Dann gilt offenbar der Satz:

Eine LINKS-*paratalktische Congruenz enthält alle* RECHT*seitigen Erzeugenden der absoluten Fläche.*

Diese werden dargestellt durch die Gleichungen

$$X_1 = X_3 = X_5 = 0, \quad X_2^2 + X_4^2 + X_6^2 = 0. \tag{20, r}$$

Ebenso enthält eine rechts-parataktische Congruenz alle *link*seitigen Erzeugenden

$$X_1^2 + X_3^2 + X_5^2 = 0, \quad X_2 = X_4 = X_6 = 0. \tag{20, l}$$

Die imaginären Leitlinien einer linkseitigen (*rechtseitigen*) *parataktischen Congruenz sind also linkseitige* (*rechtseitige*) *Erzeugende der absoluten Fläche.* Ihre Coordinaten lassen sich ebenfalls ohne Weiteres bestimmen, sollen uns aber hier nicht beschäftigen.

Schliesslich wird der Vollständigkeit halber noch zu bemerken sein:

Zwei Speere (*oder die entsprechenden Geraden*) *schneiden einander, wenn entweder* ang (Y, Z) *oder* dist $(Y, Z) \equiv 0$ mod. π ist.

Sie sind zu einander orthogonal, das heisst, jeder schneidet die absolute Polare des anderen, wenn entweder ang (Y, Z) *oder* dist $(Y, Z) \equiv \frac{\pi}{2}$ mod. π ist.

In beiden Fällen schneiden sich zu zweien die Geraden der zugehörigen Linienkreuze.

Auf den letzten Entwickelungen beruhen die Begriffe, die man, wie bemerkt, zwar nicht genügend erklärt, aber durch die Worte „linksgewunden" und „rechtsgewunden" wenigstens angedeutet hat. Denken wir uns zunächst einen Speer X und eine Gerade Y, die weder den Speer X selbst noch auch dessen absolute Polare treffen soll. Dann geht durch jeden Punkt y von Y eine Secante an x und an die absolute Polare von X. Bewegt sich y auf Y, so werden sich die Spuren dieser Secante in irgend einem Sinne auf X und auf der Polare von X bewegen. Erfolgen beide Bewegungen in gleichem, z. B. in positivem Sinne, so werden wir sagen, Y sei in Bezug auf X *rechtsgewunden*, andernfalls *linksgewunden*. Aber diese Eigenschaften ändern sich nicht, wenn man den Speer X und folglich auch seine Polare umkehrt. Daher handelt es sich um eine Lagenbeziehung zwischen den beiden *Geraden* X, Y. Stellt man das zugehörige analytische Kriterium auf, so sieht man, dass überdies auch diese beiden Geraden ihre Rolle wechseln können, ohne dass die Eigenschaft, links- oder rechts-gewunden zu sein, verloren ginge. Es gilt nämlich der Satz:

Zwei reelle Gerade Y, Z, die einander nicht schneidenden Linienkreuzen angehören, sind (jede in Bezug auf die andere) rechtsgewunden, wenn

$$(YZ)_l^2 > (YZ)_r^2, \tag{21r}$$

sie sind linksgewunden, wenn

$$(YZ)_l^2 < (YZ)_r^2. \tag{21l}$$

Die geometrische Bedeutung dieser Ungleichungen liegt auf der Hand.

Der Grenzfall $(YZ)_l^2 = (YZ)_r^2$ vermittelt natürlich den Uebergang zwischen beiden Arten der Windung. Insbesondere folgt:

*Je zwei links-parataktische Gerade, die nicht demselben Linienkreuz angehören, sind rechtsgewunden, und je zwei rechts-parataktische Gerade entsprechend linksgewunden.**)

Das Vorgetragene wollen wir nun auf die Kinematik anwenden, und zwar zunächst auf infinitesimale Bewegungen im elliptischen Raume. Eine solche Bewegung wird, wenn sie keine Schiebung ist, infinitesimale Bewegungen beider Bildkugeln zugleich bewirken. Jede dieser beiden Bewegungen kann aber aufgefasst werden als Drehung um einen *eindeutig* bestimmten Pol (X_l, X_r), die im *positiven* Drehungssinne erfolgt. Sind $2\Theta'_l$ und $2\Theta'_r$ die — nach Voraussetzung positiven — Winkelgeschwindigkeiten der beiden Drehungen, die wir uns „quantitativ-bestimmt“ — innerhalb eines sogenannten Zeitelementes *dt* ausgeführt — denken wollen, so ergeben sich (aus der Formel Nr. 17) im elliptischen Raume zwei entsprechende Grössen $2\vartheta'$ und $2\eta'$, die wir als *Drehungsgeschwindigkeit* und *Gleitungsgeschwindigkeit* der infinitesimalen Bewegung bezeichnen wollen, aus den Gleichungen

$$\Theta'_l = \vartheta' + \eta', \quad \Theta'_r = \vartheta' - \eta' \tag{22}$$

Und zwar ist $2\eta'$ die auch dem Vorzeichen nach bestimmte Geschwindigkeit, mit der irgend ein Punkt auf dem Bilde X des Polpaares X_l, X_r fortschreitet, und $2\vartheta'$ ebenso die Geschwindigkeit, mit der irgend ein Punkt auf der absoluten Polare von X fortschreitet. Die infinitesimale Bewegung selbst erscheint als eine — ebenfalls quantitativ — bestimmte „Schraubung" um den Speer X. Ihre sogenannte Wiederholung erzeugt eine gleichförmige Schraubenbewegung mit constanter Drehungs- und Gleitungsgeschwindigkeit.

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*) Durch das Gesagte ist mittelbar ein „Windungssinn“ erklärt für jede reelle nicht ebene oder conische geradlinige Fläche, insbesondere für die Tangentenfläche einer Raumcurve (und damit auch für diese selbst), soweit es sich um Stellen allgemeiner Lage handelt, nämlich um solche, an denen zwei „consecutive“ Erzeugende einander nicht schneiden. (Vergleiche die Anmerkung auf Seite 140).

Jede (quantitativ bestimmte reelle) infinitesimale Bewegung, die keine Schiebung ist, und jede solche gleichförmige Bewegung, kann als Schraubung um einen EINDEUTIG *bestimmten Speer aufgefasst werden. Es kommt ihr eine auch dem Vorzeichen nach bestimmte Drehungs- und Gleitungsgeschwindigkeit zu* *).

Diese beiden Grössen $2\vartheta'$ und $2\eta'$ genügen der Ungleichung

$$\vartheta' > |\eta'|. \tag{23}$$

Die gleichförmige Bewegung ist *periodisch*, wenn $\vartheta' : \eta'$ eine rationale Zahl ist. Sie reducirt sich auf eine *Drehung*, wenn $\eta' = 0$ ist. Beide Bildkugeln rotiren dann mit der Geschwindigkeit $2\vartheta'$. Der Begriff der Schraubung umfasst also, nach unserer Erklärung, den Begriff der Drehung. — Hierzu gehört als Ergänzung:

Jede (quantitativ bestimmte reelle) infinitesimale Schiebung und jede gleichförmige (nicht identische) Schiebung kann auf ∞^2 Arten als Schraubung um einen Speer aufgefasst werden.

Alle diese Speere bilden eine EINDEUTIG *bestimmte syntaktische Congruenz, und sie alle gehören zu derselben Drehungsgeschwindigkeit $2\vartheta'$ und Winkelgeschwindigkeit $2\eta'$.*

Ist die Schiebung linkseitig, so ist $\vartheta' - \eta' = 0$, ist sie rechtseitig, so ist $\vartheta' + \eta' = 0$.

Die zugehörige Speercongruenz ist im ersten Falle linkseitig, im zweiten rechtseitig.

Lässt man z. B. bei gegebener Schraubenaxe X η' von Null an wachsen, so giebt es im Augenblicke $\eta' = \vartheta'$ noch ∞^2 weitere Axen, darunter die absolute Polare des Speeres X. Wächst η' weiter, so vertauscht es seine Rolle mit ϑ', und die Axe ist nicht mehr der Speer X, sondern dessen absolute Polare.

Analytisch wird man eine infinitesimale Bewegung am Einfachsten dadurch ausdrücken, dass man den früher eingeführten Parametern a_i, b_i folgende specielle Werthe beilegt:

$$\begin{aligned} a_0 &= 1,\ a_1 = \alpha_1\,dt,\ a_2 = \alpha_2\,dt,\ a_3 = \alpha_3\,dt, \\ b_0 &= 1,\ b_1 = \beta_1\,dt,\ b_2 = \beta_2\,dt,\ b_3 = \beta_3\,dt. \end{aligned} \tag{24}$$

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*) Der Begriff der „qualitativ-bestimmten" infinitesimalen Bewegung entsteht durch Elimination des Zeitelementes dt. Man kann dann nur noch von einem Verhältniss $\vartheta' : \eta'$ reden.

Nach der bisher geltenden Auffassung hat die infinitesimale Bewegung *zwei* Axen, die Gerade (nicht Speere) sind, und nicht weiter unterschieden werden.

Unter dt wird hier eine Grösse verstanden, deren Quadrat gegen dt selbst vernachlässigt wird, und α_i, β_i bedeuten Grössen, die nicht alle zugleich verschwinden. Liegt dann nicht gerade eine Schiebung vor, so ergiebt sich nach einiger Rechnung

$$\Theta_l' = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}, \quad \Theta_r' = \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}, \tag{25}$$

wo beiden Wurzelgrössen die *positiven* Werthe beizulegen sind, und

$$\begin{aligned} -X_0 &= \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} . \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}, \\ X_1 &= \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} . \alpha_1, \qquad \text{u. s. w.,} \\ X_2 &= \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} . \beta_1, \qquad \text{u. s. w.} \end{aligned} \tag{26}$$

Handelt es sich um eine z. B. linkseitige Schiebung, so treten an Stelle der Gleichungen (24, 25) die folgenden:

$$\begin{aligned} &a_0 = 1, \ a_1 = \alpha_1 dt, \ a_2 = \alpha_2 dt, \ a_3 = \alpha_3 dt, \\ &b_0 = 1, \ b_1 = 0, \quad b_2 = 0, \quad b_3 = 0, \end{aligned} \tag{24l}$$

$$\Theta_l' = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}, \ \Theta_r' = 0, \tag{25l}$$

während (26) der Form nach ungeändert bleibt, aber nunmehr die Grössen $\beta_1 : \beta_2 : \beta_3$ als willkürliche Parameter enthält.

Der Satz, dass jede infinitesimale Bewegung als *Schraubung* aufgefasst werden kann, lässt sich natürlich ohne Weiteres auf *endliche Bewegungen* ausdehnen. Doch scheint es uns nicht zweckmässig (weil nur innerhalb gewisser Grenzen möglich) auch hier unter den möglichen Schraubenaxen eine bestimmte Auswahl zu treffen.

Denken wir uns (was ohne Weiteres geschehen kann) eine durch eine orthogonale Substitution gegebene Bewegung in zwei verschiedenartige Schiebungen mit Parametern a_i, b_i zerlegt, so erhalten wir zunächst zwei Drehungen der Bildkugeln. Wir wollen annehmen, dass keine von diesen sich auf die identische Substitution reducirt. Auf der linken wie auf der rechten Bildkugel giebt es dann ein Paar von Punkten X_l und X_r, die in Ruhe bleiben. Wir treffen unter ihnen, und den vier zugehörigen Speeren im elliptischen Raume, irgend eine Auswahl, indem wir in den Gleichungen

$$\begin{aligned} -X_0 &= \sqrt{a_1^2 + a_2^2 + a_3^2} . \sqrt{b_1^2 + b_2^2 + b_3^2}, \\ X_1 &= \sqrt{b_1^2 + b_2^2 + b_3^2} . a_1 \qquad \text{u. s. w.,} \\ X_2 &= \sqrt{a_1^2 + a_2^2 + a_3^2} . b_1 \qquad \text{u. s. w.} \end{aligned} \tag{27}$$

den vorkommenden Wurzelgrössen *irgend welche* zulässigen Werthe beilegen. Die zu den Drehungen der Bildkugel gehörigen Drehungswinkel $2\Theta_l$, $2\Theta_r$ sind dann bis auf Vielfache von 2π, sonst aber vollkommen, bestimmt durch die Gleichungen

$$\cos\Theta_l = \frac{a_0}{\sqrt{a_0^2+a_1^2+a_2^2+a_3^2}}, \qquad \sin\Theta_l = \frac{\sqrt{a_1^2+a_2^2+a_3^2}}{\sqrt{a_0^2+a_1^2+a_2^2+a_3^2}}, \tag{281}$$

und durch die entsprechenden Gleichungen für Θ_r. Hieraus und aus den Gleichungen

$$\Theta_l = \vartheta + \eta, \quad \Theta_r = \vartheta - \eta \tag{29}$$

ergeben sich dann der *Drehungswinkel* 2ϑ und die *Gleitungsgrösse* 2η der Schraubung um den Speer X: Diese beiden sind nur bestimmt bis auf Vielfache von π, aber — was wohl zu beachten ist — in der Weise, dass *ungerade* Vielfache von π nur zu beiden *gleichzeitig* hinzugefügt werden können.

Durch Auflösung der angeführten Gleichungen erhält man umgekehrt *die Parameter einer Bewegung, wenn der Speer X als Schraubenaxe und ausserdem* — natürlich auch dem Vorzeichen nach — *der Drehungswinkel 2ϑ und die Gleitungsgrösse 2η gegeben sind*:

(30)

$$\begin{aligned} a_0 : a_1 : a_2 : a_3 &= -X_0 \operatorname{ctg}(\vartheta+\eta) : X_1 : X_3 : X_5, \\ b_0 : b_1 : b_2 : b_3 &= -X_0 \operatorname{ctg}(\vartheta-\eta) : X_2 : X_4 : X_6.{}^{*} \end{aligned}$$

Welche Aenderungen an einigen dieser Formeln anzubringen sind, wenn die vorgelegte Bewegung in dem bisher ausgeschlossenen Falle sich auf eine Schiebung reducirt, ist ohne Weiteres zu erkennen.

Jede endliche Bewegung kann, wie man sich auszudrücken pflegt, durch eine infinitesimale Bewegung „erzeugt" werden, und zwar immer auf unendlich viele Arten.

Zunächst nämlich kann man, im Allgemeinen auf vier Arten, im Falle einer nicht identischen Schiebung auf $2.\infty^2$, und im Falle der identischen Trans-

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*) Vielleicht schreibt man diese Formeln besser noch so:

$$a_0 = -X_0 \cos(\vartheta+\eta), \; a_1 = X_1 \sin(\vartheta+\eta), \ldots \text{ (u. s. w.)}$$

(wo dann $\Sigma a^2_i = \Sigma b_i^2$ wird).

formation auf ∞^4 Arten, einen Speer X bestimmen, um den die Schraubung erfolgen soll. Sodann hat man in den Formeln

$$\begin{gathered}(31)\\ a_0' : a_1' : a_2' : a_3' = -X_0 \,.\, \mathrm{ctg}\,\{(\Theta_l + \varkappa_l \pi)\, t\} : X_1 : X_3 : X_5,\\ b_0' : b_1' : b_2' : b_3' = -X_0\, \mathrm{ctg}\,\{(\Theta_r + \varkappa_r \pi)\, t\} : X_2 : X_4 : X_6,\end{gathered}$$

worin $\varkappa_l$ und $\varkappa_r$ irgend welche ganze Zahlen bedeuten, die Parameter einer *eingliedrigen Gruppe* von Bewegungen vor sich, wie man sich sofort überzeugen kann. Zwei Transformationen dieser Gruppe werden dadurch nach einander ausgeführt, dass man die entsprechenden Werthe des Parameters t addirt. Setzt man $t = 1$, so reduciren sich die Gleichungen (31) auf die Gleichungen (30), man erhält gerade die gegebene Bewegung, und zwar bei jeder beliebigen Wahl der ganzen Zahlen $\varkappa_l$, $\varkappa_r$. Dabei liefern (von dem etwas verwickelteren Fall der Schiebungen abgesehen) jedesmal vier solche Formelsysteme dieselbe Gruppe: Nur bei einer von diesen vier Darstellungsformen erscheint die erzeugende infinitesimale Bewegung in der ausgezeichneten Gestalt, die wir im vorigen Absatz erörtert haben. *Jede* eingliedige Gruppe aber, die die gegebene Bewegung enthält, wird von den abgeleiteten Formeln geliefert. Eine besondere Erwähnung verdient der Fall $\Theta_l \equiv \Theta_r \equiv \frac{1}{2}\pi \bmod. \pi$. Die Bewegung ist dann eine *Umwendung*. Diese kann, unter Anderem, als Drehung durch den Winkel π um X, aber auch als Drehung durch denselben Winkel um die Polare von X aufgefasst werden. Zwei solche Drehungen hinter einander ausgeführt ergeben also die identische Transformation: Hierin liegt der Grund für die oben hervorgehobene besondere Art der Unbestimmtheit der Grössen 2ϑ, 2η.

Allgemein ist zu bemerken, dass eine der eingliedrigen Gruppen, die eine vorgelegte Bewegung umfassen, algebraisch (und zwar rational) wird, wenn eines der Verhältnisse $\Theta_l : \Theta_r$ (oder $\vartheta : \eta$) eine rationale Zahl ist, und dass sie alle algebraisch sind, wenn die Verhältnisse $\Theta_l : \Theta_r : \pi$ rationale Zahlen sind.

Um eine folgende Untersuchung vorzubereiten, fügen wir noch einige Bemerkungen hinzu über den Begriff, der dem Begriff einer gewöhnlichen Schraubenfläche im Euklidischen Raume analog ist, und demnach von uns durch dasselbe Wort bezeichnet werden soll.*)

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*) Dieser Abschnitt kann zunächst übergangen werden.

Es sei der Speer X die Axe, oder eine Axe, einer eingliedrigen Gruppe von Bewegungen. Wenn dann die Gerade Y den Speer X, und also auch dessen absolute Polare senkrecht schneidet, so nimmt sie bei den Bewegungen der Gruppe ∞^1 Lagen an, deren Inbegriff wir *Schraubenfläche* nennen. Es ist klar, dass alle Schraubenflächen zu einander congruent sind, deren erzeugende eingliedrige Gruppen diese Eigenschaft haben. Charakteristisch für die (gegenüber Bewegungen) invarianten Eigenschaften einer Schraubenfläche ist also das Verhältniss $\Theta_l'^2 : \Theta_r'^2$. Ist diese Grösse von 0 und ∞ verschieden, so giebt es *vier* demselben Linienkreuz angehörige Speere X, die als *Axen* der Schraubenfläche bezeichnet werden können, und, abgesehen vom Falle $\Theta_l'^2 = \Theta_r'^2$, ∞^5 Schraubenflächen, die zu der gegebenen congruent sind. Unter den vier Speeren wird einer ausgezeichnet durch die Forderung, dass $\Theta'_l > 0$, $\Theta'_r > 0$ sein soll. Die Schraubenfläche ist *rechtsgewunden*, wenn $\Theta_l'^2 < \Theta_r'^2$, *linksgewunden*, wenn $\Theta_l'^2 > \Theta_r'^2$. Das heisst, je zwei consecutive Erzeugende der Schraubenfläche sind rechts- oder linksgewunden. (S. 134). Im Grenzfall $\Theta_l'^2 = \Theta_r'^2$ ist die Schraubenfläche ein unendlich oft überdecktes Büschel. Ist dagegen das genannte Verhältniss gleich 0 oder gleich ∞, so giebt es zwei durch den Umkehrungsprocess in einander übergehende Schaaren von je ∞^1 syntaktischen Speeren, die sämmtlich „Axen“ der Fläche sind. Die geradlinigen Träger dieser Speere bilden eine zweite Schraubenfläche, vom entgegengesetzten Windungssinn. Beide Schaaren von ∞^1 Geraden liegen auf einer Fläche zweiter Ordnung, einer speciellen sogenannten *Clifford'*schen Fläche (die, wie übrigens alle Schraubenflächen, zugleich Minimalfläche ist, und bei allen Schraubungen einer zweigliedrigen Gruppe in Ruhe bleibt). Solcher specieller Schraubenflächen giebt es nur ∞^4 Individuen, und alle z. B. linksgewundenen — durch linkseitige Schiebungen erzeugten — unter ihnen sind zu einander congruent. Jede der ∞^4 Clifford'schen Minimalflächen ist Ort aller Punkte, die von den Geraden eines Linienkreuzes gleiche Abstände ($\equiv \frac{1}{4}\pi$ mod. π) haben.

Nach Auswahl der orientirten Axe X kommt jeder Schraubenfläche eine bestimmte *Ganghöhe* zu, gegeben durch den Ausdruck

$$\frac{\eta'}{\vartheta'} \cdot 2\pi = \frac{\Theta_l' - \Theta_r'}{\Theta_l' + \Theta_r'} \cdot 2\pi, \tag{32}$$

dessen geometrische Bedeutung evident ist.

Die Ganghöhe ist also positiv bei linksgewundenen, negativ bei rechtsgewundenen

*Schraubenflächen.**) Steht sie zur Zahl π in einem rationalen Verhältniss, so ist die Schraubenfläche geschlossen und algebraisch (überdies auch rational). Die absolute Polare einer Schraubenfläche ist eine zu ihr congruente Schraubenfläche. Beide Flächen sind identisch, wenn sie algebraisch sind, und überdies ϑ' und η' sich verhalten wie zwei *ungerade* Zahlen. Ein Paar absolut-polarer Schraubenflächen aber, mögen sie verschieden sein oder nicht, kann als *Ort von Linienkreuzen* angesehen werden: Man kann also auch diese zu „Schraubenflächen" zusammenfassen. Endlich gehen aus einer gewöhnlichen Schraubenfläche durch den Orientirungsprocess zwei als *Oerter von Speeren* betrachtete „Schraubenflächen" hervor. Diese haben analytischen Zusammenhang, und sind also identisch, wenn sie algebraisch sind und überdies in der reducirten Form des Quotienten $\eta' : \vartheta'$ der Zähler oder Nenner eine von Null verschiedene *gerade* Zahl ist. Ausserdem müssen wir sie wohl auch als nicht-verschieden erachten, wenn η' oder ϑ' den Werth Null hat.

Um zwei verschiedene gerade Linien Y, Z auf alle möglichen Arten durch Schraubenflächen zu verbinden, bestimme man zuerst alle Speere X, die Y und Z senkrecht treffen. Ist X ein solcher Speer, und sind Y und Z durch Orientirung ebenfalls in Speere verwandelt, so sind dann zunächst zwei Winkelgrössen $2\Theta_l$, $2\Theta_r$ bis auf Vielfache von 2π bestimmt:

$$2\Theta_l \equiv (Y_l,\ Z_l),\quad 2\Theta_r \equiv (Y_r,\ Z_r),\ \text{mod. } 2\pi.$$

Sodann kann man einen einzigen X und Y senkrecht schneidenden Speer H finden, so dass

$$\frac{\pi}{2} \equiv (Y_l,\ H_l),\quad \frac{\pi}{2} \equiv (Y_r,\ H_r),\ \text{mod. } 2\pi.$$

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*) Hier ergiebt sich, beim Grenzübergang zum Euklidischen Raume, eine Nichtübereinstimmung mit der üblichen Terminologie, die den „rechts"-gewundenen Schraubenflächen eine positive Ganghöhe beilegt. Der Grund hierfür liegt in einer Inconsequenz dieser Terminologie, die hier zur Sprache gebracht werden muss, weil sonst unsere Definitionen der Begriffe linksgewunden und rechtsgewunden als unzweckmässig erscheinen würden.

Orientirt man das Coordinatensystem so, wie auf Seite 127 beschrieben, so erhält—nach der üblichen Terminologie—eine rechtsgewundene Schrauben*linie* eine positive Ganghöhe. Ein Paar von rechts-gewundenen Geraden liegt dann so zu einander, wie eine Tangente dieser Schraubenlinie und ihre Axe; ebenfalls in Übereinstimmung mit den Festsetzungen des Textes sowohl als den üblichen. (S. z. B. *Zindler*, Liniengeometrie, I, §1). Dann aber muss man, wie uns scheint, eine geradlinige Fläche „rechtsgewunden" nennen, wenn oder wo zwei consecutive Erzeugende der Fläche rechtsgewunden sind. Daraus ergiebt sich, dass die gewöhnliche Schrauben*fläche*, auf der die rechtsgewundene Schraubenlinie liegt, als *linksgewunden* bezeichnet werden muss, während sie gewöhnlich „rechtsgewunden" heisst. Dagegen ist die Tangentenfläche einer rechtsgewundenen Schraubenlinie ebenfalls rechtsgewunden. Umgekehrt wird allgemein eine Raumcurve mit rechtsgewundener Tangentenfläche als rechtsgewunden zu bezeichnen sein.

Bewirkt man dann noch, dass $Y_0 = H_0$ wird, so bilden die ∞^1 Geraden mit den Coordinaten

$$Y_1 \cos 2\,(\Theta_l + \varkappa_l \pi)\,t + H_1 \sin 2\,(\Theta_l + \varkappa_l \pi)\,t,$$
$$Y_2 \cos 2\,(\Theta_r + \varkappa_r \pi)\,t + H_2 \sin 2\,(\Theta_r + \varkappa_r \pi)\,t,$$

u. s. w. (worin $\varkappa_l$ und $\varkappa_r$ ganze Zahlen bedeuten), oder die entsprechenden Speere mit der siebenten Coordinate $\pm\sqrt{2}\,Y_0 = \pm\sqrt{2}\,H_0$ eine Schraubenfläche, die für t = 0 die Gerade oder den Speer Y, und für t = 1 die Gerade oder den Speer Z liefert, und jede Schraubenfläche durch Y, Z kann so gefunden werden.

Man sieht, dass in der Regel zwischen verschiedenen Schraubenflächen durch Y, Z kein continuirlicher Uebergang möglich ist. Ebenso aber übersieht man auch leicht das verwickeltere Verhalten, das dann eintritt, wenn die Geraden oder Speere Y, Z parataktisch sind, oder wenn sie die noch speciellere Figur von zwei Geraden desselben Linienkreuzes bilden.

Nachdem man zur eindeutigen Darstellung der reellen Speere durch reelle Punkte zweier Kugeln gelangt ist, liegt es nahe, die ganze Betrachtung zu vertiefen. Zunächst wird man sie auf das complexe Gebiet auszudehnen suchen. Sodann wird man umfassendere Gruppen von Transformationen und weitere Aequivalenzbegriffe einführen, und zwar zunächst solche Transformationen, bei denen beide Kugeln irgendwie collinear tranformirt (und eventuell auch noch vertauscht) werden.

Setzen wir die Begriffe Gerade und Speer mit Hülfe ihrer Coordinaten ins complexe Gebiet hinein fort, so wird eine *complexe Gerade* durch sechs Verhältnissgrössen X_i definirt, die der Gleichung (4) genügen, und möglicher Weise imaginäre Verhältnisse haben. Analog wird ein *complexer Speer* definirt durch sieben complexe (reelle oder imaginäre) Verhältnissgrössen, die den beiden Gleichungen (5) genügen. Man sieht daraus, dass das Continuum der complexen Speere das Continuum der complexen Geraden in der Weise doppelt überdeckt, dass die Tangenten der absoluten Fläche, nämlich die Geraden des Complexes

$$X_1^2 + X_3^2 + X_5^2 = 0, \ X_2^2 + X_4^2 + X_6^2 = 0 \tag{33}$$

als *Verzweigungselemente* fungiren: Diese Geraden sind zugleich auch Speere, und jeder solche *Minimalspeer* ist nicht zu unterscheiden von dem umgekehrten Speer, da ein Vorzeichenwechsel von X_0 bedeutungslos wird, wenn X_0 verschwindet.

Andrerseits können wir auch „complexe Punkte“ unserer beiden Bildkugeln betrachten: Diese werden erschöpfend dargestellt durch zwei Systeme homogener complexer Cooadinaten $\mathfrak{x}_{l0} : \mathfrak{x}_1 : \mathfrak{x}_3 : \mathfrak{x}_5$ und $\mathfrak{x}_{r0} : \mathfrak{x}_2 : \mathfrak{x}_4 : \mathfrak{x}_6$, die verbunden sind durch Gleichungen der Form

$$\mathfrak{x}_{l0}^2 = \mathfrak{x}_1^2 + \mathfrak{x}_3^2 + \mathfrak{x}_5^2, \qquad \mathfrak{x}_{r0}^2 = \mathfrak{x}_2^2 + \mathfrak{x}_4^2 + \mathfrak{x}_6^2 \tag{34}$$

Nun sieht man sogleich, dass der nachgewiesene *Zusammenhang* zwischen reellen Speeren und reellen Punktepaaren beider Kugeln sich *nicht* vollständig auf die complexen Gebiete übertragen lässt; denn die Gleichungen

$$\mathfrak{x}_{l0} : \mathfrak{x}_1 : \mathfrak{x}_3 : \mathfrak{x}_5 = X_0 : X_1 : X_3 : X_5, \qquad \mathfrak{x}_{r0} : \mathfrak{x}_2 : \mathfrak{x}_4 : \mathfrak{x}_6 = X_0 : X_2 : X_4 : X_6, \tag{35}$$

die den Zusammenhang zwischen den Punktepaaren $\mathfrak{x}_l$, $\mathfrak{x}_r$ und den Speeren X vermitteln, lassen sich nach den Grössen X_i überhaupt nicht auflösen, wenn $\mathfrak{x}_{l0} = 0$, $\mathfrak{x}_{r0} \neq 0$ oder $\mathfrak{x}_{l0} \neq 0$, $\mathfrak{x}_{r0} = 0$ ist, und sie lassen sich nicht eindeutig auflösen, wenn $\mathfrak{x}_{l0}$ und $\mathfrak{x}_{r0}$ zugleich verschwinden.

Hieraus geht nun hervor, dass der Begriff des Speeres (mindestens) *zwei* natürliche Fortsetzungen ins complexe Gebiet zulässt: Man kann einmal, wie gesagt, die durch die Gleichung (5) verbundenen Grössen X_i als Coordinaten eines „complexen Speeres“ ansehen, dann aber auch die durch die Gleichungen (34) verbundenen doppelt-homogenen Grössen $\mathfrak{x}_i$. Welches auch der noch festzustellende begriffliche Inhalt der zweiten Erweiterung sein möge, wir erhalten ein Continuum complexer Speere, das ganz andere Eigenschaften haben muss, als das zuerst genannte. Für verschiedene Dinge aber braucht man, wenn keine Unklarheit entstehen soll, verschiedene Worte. Wir werden daher die „Speere“ des zweiten Continuums nun überhaupt nicht mehr Speere, sodern *Pfeile* nennen.

Ein Pfeil wird also dargestellt durch zweimal vier homogene Coordinaten, die durch die beiden Gleichungen (34) *verbunden sind.**)

Die Begriffe Speer und Pfeil decken einander im reellen Gebiete vollständig, im complexen Gebiete aber nur zum Theil.

Denken wir uns nun z. B. die Punkte

$$\mathfrak{x}_{l0} : \mathfrak{x}_1 : \mathfrak{x}_3 : \mathfrak{x}_5$$

der linken Bildkugel irgend einer collinearen Transformation unterworfen, die

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*) Wir anticipiren hier die fernere Entwickelung. Eigentlich müssten wir sagen: Der Pfeil *ist* das System seiner Coordinaten.

die erste der Gleichungen (34) bestehen lässt, so gehen aus ebenen Schnitten, also Kreisen, wieder solche hervor. Wir haben die Gruppe der *Mölius*'schen Kreisverwandtschaften vor uns, die auf der Kugel gedeutet werden. Wir wollen diese Transformationen *cyclische* Transformationen der Kugel nennen. Verfahren wir ebenso mit der zweiten Kugel, und fügen wir schliesslich auch noch collineare Vertauschungen beider Kugeln hinzu, (z. B. durch die früher betrachteten Bewegungen), so erhalten wir eine Gruppe, die insgesammt *acht* continuirliche Schaaren von Transformationen umfasst, da schon jede der beiden cyclischen Gruppen „gemischt“ ist, nämlich aus zwei Schaaren von „eigentlichen“ und „uneigentlichen“ cyclischen Transformationen besteht. Alle diese Transformationen, $8 . \infty^{12}$ im reellen, und $8 . \infty^{24}$ im complexen Gebiete, können als eindeutige Transformationen der Pfeile, *nicht* aber auch der Speere aufgefasst werden: Schon eine *reelle* cyclische Transformation beider Kugeln bewirkt in der Regel nicht eine durchweg-eindeutige Transformation der *complexen* Speere.

Wir werden also die gefundenen Vertauschungen die Pfeile nunmehr *cyclische Transformationen der Pfeile* nennen; und insbesondere werden wir durch das Epitheton *eigentlich* die unter ihnen auszeichnen, die durch Zusammensetzung einer eigentlichen cyclischen Transformation der linken Kugel mit einer eigentlichen cyclischen Transformation der rechten Kugel entstehen. Die *reellen* eigentlichen cyclischen Transformationen der Pfeile bilden dann eine reell-continuirliche Gruppe von ∞^{12} Transformationen, deren jede auf eine einzige Weise in eine *linkseitige cyclische Schiebung* (bei der alle Punkte der rechten Kugel einzeln in Ruhe bleiben) und eine mit dieser vertauschbare *rechtseitige cyclische Schiebung* zerlegt werden kann. Die reellen linkseitigen cyclischen Schiebungen z. B. bilden für sich eine reell-einfache*) sechsgliedrige invariante Untergruppe der Gruppe der reellen eigentlichen cyclischen Transformationen. Die *complexen* eigentlichen cyclischen Transformationen dagegen können analog auf eine einzige Weise in *vier* mit einander vertauschbare Transformationen zerlegt werden, da jede eigentliche collineare oder cyclische Transformation z. B. der linken Bildkugel noch in zwei Transformationen (Schiebungen der Kugel) zerlegt werden kann, deren jede alle Geraden der einen Schaar von Erzeugenden dieser Kugel einzeln in Ruhe lässt. Jede der vier genannten Arten specieller Transformationen bildet wiederum eine invariante einfache Untergruppe mit sechs (nach der üblichen nicht ganz einwandsfreien Terminologie drei) wesentlichen Parametern.

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*) Eine Gruppe heisst einfach, wenn sie keine invariante Untergruppe hat.

Näher wollen wir die bemerkenswerthe Gruppe der cyclischen Pfeiltransformationen hier nicht betrachten; wir wollen aber einige ihrer nächstliegenden Eigenschaften zusammenstellen, um es deutlich hervortreten zu lassen, wie sich die elliptische Geometrie in einen umfassenderen Gedankenkreis einordnet.*)

Zunächst heben wir hervor:

Die cyclischen Transformationen der Pfeile sind in dem Continuum aller Pfeile überall wohldefinirt, eindeutig und stetig.

Im reellen, nicht aber durchweg im complexen Gebiete, bewirken sie auch eineindeutige Transformationen der Speere (da deren Begriff hier mit dem der Pfeile zusammenfällt) und also zwei-zweideutige Transformationen der geraden Linien.

Da die Begriffe Syntaxie, Antitaxie, Umkehrung ohne Weiteres auf die Pfeile übertragen werden können, so können wir ferner sagen:

Die ∞^{12} (∞^{24}) eigentlichen cyclischen Pfeiltransformationen lassen aus links- (rechts-) syntaktischen Pfeilen immer wieder ebensolche hervorgehen.

Die Eigenschaft zweier Pfeile antitaktisch zu sein, wird dagegen durch diese Transformationen in der Regel zerstört. Die Invarianz auch dieser Eigenschaft kennzeichnet die sechsgliedrige Untergruppe der Bewegungen im elliptischen Raume.

Man kann dies offenbar auch so ausdrücken:

Die Bewegungen im elliptischen Raume umfassen alle eigentlichen cyclischen Transformationen der Pfeile, die mit der Umkehrung vertauschbar sind, und deshalb auch als eindeutige Transformationen gerader Linien aufgefasst werden können.

Ferner ist evident:

Je drei reelle Pfeile, deren keine zwei syntaktisch sind, lassen sich der Reihe nach in drei andere gleicher Eigenschaft durch eine einzige eigentliche cyclische Pfeiltransformation überführen.

Vier solche Pfeile haben dem entsprechend *vier* unabhängige absolute Invarianten, die sich leicht durch Doppelverhältnisse darstellen lassen†). Ebenso sieht man ohne Weiteres:

Die cyclischen Transformationen der reellen Pfeile sind z. B. auch dadurch erschöpfend charakterisirt, dass sie die 2. ∞^5 Regelschaaren unter einander vertauschen, die durch Drehung je eines Pfeiles um einen zu ihm syntaktischen Pfeil entstehen.

Diese Umdrehungsörter von je ∞^1 Pfeilen sind augenscheinlich nichts Anderes als die orientirten Regelschaaren sogenannter *Clifford*'schen Flächen, deren jede vier solcher Oerter liefert. Man kann sie linkseitige und rechtseitige einfache Cyclen nennen. Es ergeben sich dann weiter Doppelcyclen, Congruenzen von Pfeilen, die, soweit sie reell sind, erschöpfend durch alle Schraubungen eines Pfeiles um einen anderen entstehen (der nicht zu demselben Linienkreuz gehören darf), u. s. w.

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*) Weitere Entwickelungen findet man in einer Abhandlung des Verfassers, Jahresber. d. Deutsch. Math. Ver. Bd. 15, 1906, S. 476–527.

†) Siehe einen Aufsatz des Verfassers, Leipz. Ber. 1896, S. 199 u. ff.

Wir betrachten nunmehr wieder nur *reelle* Pfeile oder Speere.

Die reellen Pfeile, oder was dasselbe ist, die reellen Speere lassen sich auf die beschriebene Art auf die reellen Punktepaare unserer beiden Kugeln abbilden. Die einzelnen Punkte dieser Kugeln kann man aber in bekannter Weise den Werthen von zwei complexen Veränderlichen zuordnen, die Kugeln lassen sich als sogenannte *Riemann'sche Zahlenkugeln* benutzen. So gelangen wir zu einer weiteren Art von Pfeilcoordinaten, die nichts Anderes sind als die passend gewählten binären Parameter von Erzeugenden beider Bildkugeln. Die Betrachtung dieser *Parameter eines Pfeiles,* wie wir sie nennen wollen, hat aber ein grundsätzliches Interesse, da durch sie unser Gegenstand mit Problemen der modernen Funktionentheorie in Verbindung gesetzt wird (wie in weiteren Abhandlungen näher ausgeführt werden soll). Wir führen aus diesem Grunde hier die wichtigsten Thatsachen vor, die sich auf den Zusammenhang der Pfeilcoordinaten $\mathfrak{x}_i$ und der genannten Parameter beziehen. Die eingeführte Beschränkung auf reelle Pfeile wird nachträglich sehr leicht aufgehoben werden können.

Bei der Durchführung des bezeichneten Gedankens, und bei seiner Anwendung auf die zuvor entwickelten Formeln, wird man ein- und dasselbe Formelsystem zweimal anzuwenden haben. Deshalb betrachten wir zunächst eine einzige Kugel, die wir nachher bald mit der linken bald mit der rechten unserer Bildkugeln identificiren werden. Die homogenen rechtwinkligen Coordinaten der Punkte dieser Kugel wollen wir $\mathfrak{x}_0$, $\mathfrak{x}_1$, $\mathfrak{x}_2$, $\mathfrak{x}_3$ nennen, so dass $\mathfrak{x}_0 = 0$ die unendlich ferne Ebene darstellt, und die Gleichung der Kugel so lautet:

$$\mathfrak{x}_0^2 = \mathfrak{x}_1^2 + \mathfrak{x}_2^2 + \mathfrak{x}_3^2. \tag{36}$$

Die zu dem Kugelpunkte $\mathfrak{x}$ gehörigen complexen Verhältnissgrössen erklären wir nunmehr durch die Proportion

$$\xi_1 : \xi_2 = \begin{cases} = \mathfrak{x}_0 + \mathfrak{x}_3 : \mathfrak{x}_1 + i\mathfrak{x}_2 \\ = \mathfrak{x}_1 - i\mathfrak{x}_2 : \mathfrak{x}_0 - \mathfrak{x}_3. \end{cases} \tag{37}$$

Da es bei den Coordinaten $\mathfrak{x}_i$ auf einen (hier reellen) Proportionalitätsfactor nicht ankommt, so kann man umgekehrt setzen

$$\begin{aligned} \mathfrak{x}_0 &= \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2, \\ \mathfrak{x}_1 &= \xi_1\bar{\xi}_2 + \xi_2\bar{\xi}_1, \\ -i.\ \mathfrak{x}_2 &= \xi_1\bar{\xi}_2 - \xi_2\bar{\xi}_1, \\ \mathfrak{x}_3 &= \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2. \end{aligned} \tag{38}$$

($\bar{\xi}_\kappa$ bezeichnet die zu ξ_κ conjugirt-complexe Grösse).

Eine beliebige lineare Transformation der Veränderlichen $\xi_1 : \xi_2$,

$$\begin{aligned} \xi_1' &= \alpha_{11}\xi_1 + \alpha_{12}\xi_2, \\ \xi_2' &= \alpha_{21}\xi_1 + \alpha_{22}\xi_2 \end{aligned} \tag{39}$$

bewirkt jetzt eine sofort hinzuschreibende *reelle* lineare Transformation der reellen Veränderlichen $\mathfrak{x}_i$, die eine eigentliche collineare Transformation der Kugel und also eine eigentliche cyclische Transformation (eigentliche Kreisverwandtschaft) für deren Punkte zur Folge hat. Umgekehrt kann man, wenn eine eigentliche automorphe Collineation der Kugel (36) vorliegt, daraus eindeutig die Verhältnisse der Grössen $\alpha_{i\kappa}$ berechnen. Wir wollen diese Formeln nicht explicite aufstellen, wollen aber von ihnen eine Anwendung machen. Wir wollen nämlich annehmen, es sei insbesondere eine reelle Bewegung

$$\dot{\mathfrak{x}}' = \dot{\bar{a}}\,\dot{\mathfrak{x}}\,\dot{a} \tag{40}$$

vorgelegt, mit den Euler'schen Parametern $a_0 : a_1 : a_2 : a_3$, die die Kugel (36) Ruhe lässt. Dann findet sich, dass die zugehörigen Coefficienten $\alpha_{i\kappa}$ so ausgedrückt werden können:

$$\begin{aligned} &\alpha_{11} = a_0 + ia_3,\ \ \alpha_{12} = i(a_1 - ia_2), \\ &\alpha_{21} = i(a_1 + ia_2),\ \ \alpha_{22} = a_0 - ia_3.\text{*)} \end{aligned} \tag{41}$$

Die Determinante $\Delta = |\,\alpha_{i\kappa}\,|$ dieses Grössensystems hat den Werth.

$$\Delta = a_0^2 + a_1^2 + a_3^2 + a_2^2, \tag{42}$$

ist also die Norm der Quaternion $\dot{a}$. Ersetzen wir nun die Coordinaten

$$\mathfrak{x}_0,\ \mathfrak{x}_1,\ \mathfrak{x}_2,\ \mathfrak{x}_3$$

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*) Diese oder vielmehr äquivalente Formeln hat *Cayley* angegeben (Math. Ann. Bd. 15, 1879, S. 238), sie haben sich aber auch im Nachlasse von *Gauss* gefunden (Werke Bd. 8, S. 355). Vielfache Anwendung von ihnen hat Herr *F. Klein* gemacht, z. B. in seinem Buche über das Icosaeder.

im Falle der linken Bildkugel durch

$$\mathfrak{x}_{l0},\ \mathfrak{x}_1,\ \mathfrak{x}_3,\ \mathfrak{x}_5$$

und im Falle der rechten Kugel durch

$$\mathfrak{x}_{r0},\ \mathfrak{x}_2,\ \mathfrak{x}_4,\ \mathfrak{x}_6,$$

und unterscheiden wir gleichzeitig die entsprechenden Coordinatenwerthe ξ_1, ξ_2 durch Indices l, r so haben wir den Satz:

In Pfeilparametern werden die reellen linkseitigen Schiebungen dargestellt durch die Gleichungen

$$\begin{aligned} \xi'_{l1} &= (a_0 + ia_3)\,\xi_{l1} + i\,(a_1 - ia_2)\,\xi_{l2}, \\ \xi'_{l2} &= i(a_1 + ia_2)\,\xi_{l1} + (a_0 - ia_3)\,\xi_{l2}, \end{aligned} \tag{43l}$$

und

$$\xi'_{r1} = \xi_{r1},\ \xi'_{r2} = \xi_{r2};$$

ebenso die reellen rechtseitigen Schiebungen durch die Gleichungen

$$\begin{aligned} \xi'_{r1} &= (b_0 + ib_3)\,\xi_{r1} + i\,(b_1 - ib_2)\,\xi_{r2}, \\ \xi'_{r2} &= i\,(b_1 + ib_2)\,\xi_{r1} + (b_0 - ib_3)\,\xi_{r2}, \end{aligned} \tag{43r}$$

und

$$\xi'_{l1} = \xi_{l1},\ \xi'_{l2} = \xi_2.$$

Die reellen Verhältnissgrössen a_i und b_i sind die Cayley'schen Schiebungsparameter. (Vgl. S. 122, Nr. 10–12).

Der Nutzen, den die Pfeilparameter bringen, ist offenbar ein doppelter. Erstens nämlich werden die beiden quadratischen Gleichungen (34) nunmehr zu Identitäten. Es ist aber principiell einfacher, mit von einander unabhängigen Veränderlichen zu operiren, als mit solchen, die durch (wenn auch nur quadratische) Gleichungen verbunden sind. Gleichgültig hierfür ist es offenbar, ob man, wie wir es thun, die vier unabhängigen wesentlichen Veränderlichen paarweise zu sogenannten complexen Grössen zusammenfassen will, oder nicht. Diese Zusammenfassung selbst aber, die übrigens keineswegs unter allen Umständen zu empfehlen ist, bietet einen weiteren Vortheil: Es treten auf diese Weise klar die engen Beziehungen hervor, die unseren Stoff mit sonst schon als wichtig erkannten Gegenständen der Funktionentheorie und Gruppentheorie verbinden. Soweit die Bewegungsgruppe in Frage kommt, handelt es sich hierbei vor Allem um einen Zusammenhang unserer Theorie mit der Theorie der (binären) sogenannten *Hermite*'schen Formen.

Führen wir die Zeichen

$$\begin{aligned}(\xi\ \eta) &= \xi_1\eta_2 - \xi_2\eta_1\\ (\xi/\bar\eta) &= \xi_1\bar\eta_1 + \xi_2\bar\eta_2,\end{aligned}\tag{44}$$

ein, so hat jede der beiden Gleichungen $(\xi\eta) = 0$, $(\xi/\bar\eta) = 0$ eine einfache geometrische Bedeutung. Die erste sagt aus, dass die Punkte ξ, η der Kugel (36) zusammenfallen. Die zweite Gleichung aber sagt aus, dass die Punkte ξ, η einander diametral gegenüberliegen. (Vgl. Nr. 38). Das lässt sich, ganz abgesehen von der Deutung auf der Kugel, auch so ausdrücken, dass man sagt: Die Punkte ξ, η sind einander paarweise zugeordnet in dem „Polarsystem" einer (definiten) Hermite'schen Form (oder, nach *Segre,* in einer *Antinvolution* ohne Doppelelemente). Denn es ist leicht zu sehen, dass alle projectiven Spiegelungen an Punkten und Ebenen, die die Riemann'sche Kugel (36) in Ruhe lassen, mit Hülfe der binären Veränderlichen dargestellt werden können durch bilineare Gleichungen der Form

$$c_{11}\xi_1\bar\eta_1 + c_{12}\xi_1\bar\eta_2 + c_{21}\xi_2\bar\eta_1 + c_{22}\xi_2\bar\eta_2 = 0,$$

wo $c_{i\kappa} = \bar c_{i\kappa}$, $c_{11}c_{22} - c_{12}c_{21} \neq 0$ ist, und ξ, η in der Spiegelung einander zugeordnete Kugelpunkte bedeuten. Diese Gleichung aber ist, gegenüber cogredienten linearen Transformationen der Veränderlichen ξ_1, ξ_2 und η_1, η_2 invariant, verbunden mit der sogenannten Hermite'chen Form

$$c_{11}\xi_1\bar\xi_1 + c_{12}\xi_1\bar\xi_2 + c_{21}\xi_2\bar\xi_1 + c_{22}\xi_2\bar\xi_2.$$

Sie ist aus ihr durch einen Process abgeleitet, der nahe verwandt ist dem sogenannten Polarenprocess, durch den man z. B. von einer quadratischen Form aus zu deren Polarsystem kommt. Nun sieht man unmittelbar, dass bei der Transformation (41) beide Ausdrücke (44) mit dem Factor Δ reproducirt werden. Der erste Ausdruck wird auch dann noch reproducirt, wenn die Grössen $\alpha_{i\kappa}$ beliebige complexe Werthe haben (so dass $\Delta \neq 0$). Wenn aber auch der zweite diese Eigenschaft haben soll, dann müssen, wie eine leichte Rechnung zeigt, die Coefficienten $\alpha_{i\kappa}$, von einem gemeinsamen complexen Factor abgesehen, die Form (41) haben.

Alles dies haben wir zweimal anzuwenden; wir erhalten dann, bei Gebrauch der bereits erklärten Abkürzungen, die Gleichungen

$$\begin{aligned}(\xi'\eta')_l &= \Delta_l\cdot(\xi\eta)_l, & (\xi'/\bar\eta')_l &= \Delta_l\cdot(\xi/\bar\eta)_l,\\ (\xi'\eta')_r &= \Delta_r\cdot(\xi\eta)_r, & (\xi'/\bar\eta')_r &= \Delta_r\cdot(\xi/\bar\eta)_r.\end{aligned}\tag{45}$$

Die den reellen Schiebungen des elliptischen Raumes entsprechenden projectiven Transformationen der binären Gebiete (ξ_l), (ξ_r) *lassen jede ein gewisses Hermite'sches Polarsystem (eine Antinvolution) in Ruhe, und sie sind dadurch characterisirt.*

Auf jeder von beiden Bildkugeln ordnet das zugehörige Polarsystem irgend einem Punkt den diametral-gegenüberliegenden Punkt zu.

Nebensächlich ist bei diesem Satze die besondere Form der Gleichungen $(\xi/\bar\eta)=0$; diese würde sich in mannigfacher Weise ändern lassen. Wir haben offenbar, bei Aufstellung der Gleichungen (37), eine willkürliche Auswahl unter ∞^3 Möglichkeiten getroffen. Wesentlich aber ist die Zugehörigkeit dieser Gleichungen zu *definiten* Hermite'schen Formen. (Zu verlangen, dass ausser ihren Polarsystemen auch die Hermite'schen Formen selbst reproducirt werden sollen, scheint in unserem Zusammenhang nicht erforderlich und im Ganzen auch nicht zweckmässig zu sein).

Es hat sich also eine enge Beziehung der elliptischen Geometrie, und damit auch der zugehörigen Theorie der Bewegungsinvarianten, zur Geometrie und Invariantentheorie der Polarsysteme Hermite'scher Formen ergeben. Auf die allgemeinen Fragen, zu denen man von hier aus kommt, denken wir ein anderes Mal einzugehen. Hier betrachten wir nur noch einige der nächstliegenden Beziehungen: Den Invariantencharakter dieser lassen wir dadurch hervortreten, dass wir die für den vorliegenden Zweck ausreichenden Zeichen (44) systematisch verwenden.

Zunächst ergeben sich jetzt bemerkenswerthe Ausdrücke für *Winkel* und *Abstand zweier Speere Y, Z* oder $\mathfrak{y}$, $\mathfrak{z}$. Es gilt nämlich für beide Bildkugeln die Gleichung

$$(\mathfrak{y}\mathfrak{z}) = (\eta/\bar\zeta)\,(\bar\eta/\zeta) - (\eta\zeta)\,(\bar\eta\bar\zeta). \tag{46}$$

Ausserdem aber hat man die Identität

$$(\eta/\bar\eta)\cdot(\zeta/\bar\zeta) - (\eta/\bar\zeta)\,(\bar\eta/\zeta) = (\eta\zeta)\,(\bar\eta\bar\zeta). \tag{47}$$

Hieraus folgen weiter, z. B. für die linke Bildkugel

$$\left.\begin{aligned} \mathfrak{y}_{l\mathfrak{w}}\mathfrak{z}_{l\mathfrak{w}} + (\mathfrak{y}\mathfrak{z}) &= 2\,(\eta/\bar\zeta)_l\,(\bar\eta/\zeta)_l, \\ \mathfrak{y}_{l\mathfrak{w}}\mathfrak{z}_{l\mathfrak{w}} - (\mathfrak{y}\mathfrak{z}) &= 2\,(\eta\,\zeta)_l\,(\bar\eta\,\bar\zeta)_l, \end{aligned}\right\} \tag{48, l}$$

und

$$\left.\begin{aligned} \cos\tfrac{1}{2}\,(\mathfrak{y},\,\mathfrak{z})_l &= \frac{\sqrt{(\eta/\bar\zeta)_l\,(\bar\eta/\zeta)_l}}{\sqrt{(\eta/\bar\eta)_l}\,\sqrt{(\zeta/\bar\zeta)_l}} \\ \sin\tfrac{1}{2}\,(\mathfrak{y},\,\mathfrak{z})_l &= \frac{\sqrt{(\eta\zeta)_l\,(\bar\eta\bar\zeta)_l}}{\sqrt{(\eta/\bar\eta)_l}\,\sqrt{(\zeta/\bar\zeta)_l}}\;{}^{*} \end{aligned}\right\} \tag{49, l}$$

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*) Diese Formeln bilden den Ausgangspunkt für weitere Untersuchungen Math. Ann. Bd. 60, 1905, S. 321.

Die zuvor von uns aufgestellten Formeln Nr. 19 haben Cosinus und Sinus nicht der halben, sondern der ganzen Winkel $(\mathfrak{y}, \mathfrak{z})_l$ und $(\mathfrak{y}, \mathfrak{z})_r$ geliefert, haben diese aber eindeutig bestimmt. Es bleibt daher noch die Rolle der Wurzelgrössen in den Gleichungen (49) aufzuhellen.

Die Möglichkeit, über das Vorzeichen z. B. der Grösse $\sin\,(Y, Z)_l$ in den Formeln (19) zu entscheiden, entspricht nun der Möglichkeit, unter den beiden Polen des Hauptkreises durch die Punkte Y_l, Z_l einen auszuwählen. Das kann auch bei Gebrauch von Coordinaten zweiter (und dritter) Art erfolgen (vgl. Nr. 35): Man hat, entsprechend den Formeln

$$\mathfrak{x}_0 : \mathfrak{x}_1 = \sqrt{(\mathfrak{y}\mathfrak{y})_l(\mathfrak{z}\mathfrak{z})_l - (\mathfrak{y}\mathfrak{z})_l^2} : \mathfrak{y}_3\mathfrak{z}_5 - \mathfrak{y}_5\mathfrak{z}_3$$

u. s. w. einen der Wurzelwerthe

$$\begin{aligned}&\sqrt{(\mathfrak{y}\mathfrak{y})_l\,(\mathfrak{z}\mathfrak{z})_l - (\mathfrak{y}\mathfrak{z})_l^2} = \\ &= 2\,.\sqrt{(\eta/\zeta)_l\,(\bar{\eta}/\zeta)_l}\,.\sqrt{(\eta\zeta)_l\,(\bar{\eta}\bar{\zeta})_l}\end{aligned} \tag{50, 1}$$

auszuwählen. Durch Entscheidung über den Werth *des Productes* der beiden letzten reellen Wurzelgrössen wird mithin der Speer X, besser der Pfeil $\mathfrak{x}_l$, $\mathfrak{x}_r$ eindeutig festgelegt. Der Werth desselben Productes aber entscheidet auch über das Vorzeichen von $\sin\,(\mathfrak{y}, \mathfrak{z})_l$. Der Werth einer der beiden letzten Quadratwurzeln bleibt dann noch willkürlich, und lässt in Verbindung mit den beiden Wurzelgrössen im Nenner der Ausdrücke (49, 1) den Grössen $\cos\frac{1}{2}(\mathfrak{y}, \mathfrak{z})_l$ und $\sin\frac{1}{2}(\mathfrak{y}, \mathfrak{z})_l$ eben den Spielraum, den wir haben müssen, wenn wir $\sin\,(\mathfrak{y}, \mathfrak{z})_l$ als eindeutig bestimmt annehmen.

Auch die auf die Eigenschaften der Bewegungen im elliptischen Raume bezüglichen Ausdrücke gehen nun in solche über, denen man ihre Eigenschaft als Invarianten binärer Formen sogleich ansieht, wie indessen nicht weiter ausgeführt werden soll.

In das Bild, das wir hier von der Geometrie der Speere und Pfeile zu entwerfen gedachten, haben wir nun noch einen letzten aber wesentlichen Zug einzutragen.

Erinnern wir uns, dass die zu den beiden Bildkugeln gehörigen complexen Verhältnissgrössen $\xi_{l1} : \xi_{l2}$ und $\xi_{r1} : \xi_{r2}$ die Parameter von je einer Schaar von geradlinigen Erzeugenden der zugehörigen Kugel sind. Unterwerfen wir nun jede Kugel einer Bewegung, so werden diese Parameter linear transformirt. (Nr. 43). Dabei werden die auf den unendlich fernen Ebenen $\mathfrak{x}_{l0} = 0$ und $\mathfrak{x}_{r0} = 0$ gelegenen imaginären Kugelpunkte nur untereinander (projectiv) vertauscht. Diese Punkte aber sind, wie die Gleichungen (20) zeigen, eindeutig und natürlich auch gegenüber zusammengehörigen Bewegungen invariant zugeordnet den linkseitigen und rechtseitigen Erzeugenden der absoluten Fläche.

Die zur linken (rechten) Bildkugel gehörigen complexen Parameter können also auch als Parameter der linkseitigen (rechtseitigen) Erzeugenden der absoluten Fläche betrachtet werden.

Zu conjugirt-imaginären Erzeugenden der absoluten Fläche gehören diametral-gegenüberliegende Punkte der Bildkugeln, und also Parameter, die einander in den entsprechenden Hermite'schen Polarsystemen zugeordnet sind.

Die Formeln (43) *zeigen, wie die Erzeugenden beider Arten durch die reellen Bewegungen vertauscht werden.*

Ohne Weiteres ergiebt sich der Zusammenhang zwischen den Coordinaten X_ι (No. 20) und den Parametern $\xi_\varkappa$ beider Schaaren von Erzeugenden. Man hat zu diesem Zwecke nur in den Gleichungen (38) die Verhältnissgrössen $\bar{\xi}_1 : \bar{\xi}_2$ durch irgendwelche Grössen $\eta_1 : \eta_2$ zu ersetzen, und diese dann so zu bestimmen, dass $\mathfrak{x}_0 = 0$ wird; man hat also etwa an Stelle von $\bar{\xi}_1$ und $\bar{\xi}_2$ die Werthe $-\xi_2$ und ξ_1 einzutragen. Auf beide Kugeln angewendet, liefert dieses Verfahren die Coordinaten der zweierlei Erzeugenden $X_{(l)}$ and $X_{(r)}$ der absoluten Fläche, dargestellt durch die Parameter ξ:

$$\begin{aligned} X_{l1} &= \xi_{l1}^2 - \xi_{l2}^2, & X_{l2} &= 0, \\ X_{l3} &= i\{\xi_{l1}^2 + \xi_{l2}^2\}, & X_{l4} &= 0, \\ X_{l5} &= -2\,\xi_{l1}\xi_{l2}, & X_{l6} &= 0; \end{aligned} \tag{51, l}$$

$$\begin{aligned} X_{r1} &= 0, & X_{r2} &= \xi_{r1}^2 - \xi_{r2}^2, \\ X_{r3} &= 0, & X_{r4} &= i\{\xi_{r1}^2 + \xi_{r2}^2\}, \\ X_{r5} &= 0, & X_{r6} &= -2\,\xi_{r1}\,\xi_{r2}. \end{aligned} \tag{51, r}$$

Umgekehrt folgt

$$\begin{aligned} \xi_{l1} : \xi_{l2} &= X_{l1} - iX_{l3} : -X_{l5} = X_{l5} : X_{l1} + iX_{l3}, \\ \xi_{r1} : \xi_{r2} &= X_{r2} - iX_{r4} : -X_{r6} = X_{r6} : X_{r2} + iX_{r4}. \end{aligned} \tag{52}$$

Man sieht, dass bei einem Uebergang zu der conjugirt-imaginären Erzeugenden die Parameter ξ_1 und ξ_2 durch $\bar{\xi}_2$ und $-\bar{\xi}_1$ ersetzt werden.

Weiter zeigt nun eine leichte Ueberlegung, dass der reelle Speer X oder Pfeil $\mathfrak{x}_l$, $\mathfrak{x}_r$, also der Pfeil mit den Parametern $\xi_{l1} : \xi_{l2}$ und $\xi_{rl} : \xi_{r2}$, die vier soeben betrachteten Erzeugenden der absoluten Fläche schneiden muss. In der That findet sich mit Hülfe der Gleichungen (35) und (38), wenn man diese letzten nach der gegebenen Anweisung verwerthet (vgl. unten Nr. 54), dass

$$(X\Xi)_l = 0, \quad (X\Xi)_r = 0 \tag{53}$$

ist für $\Xi = X_{(l)}, X_{(r)}\ \overline{X}_{(l)}, \overline{X}_{(r)}$. (S. Nr. 51). Daher muss die Gerade des reellen Speeres X entweder den Schnittpunkt der beiden ersten unter diesen vier Erzeugenden mit dem Schnittpunkt der beiden letzten verbinden, oder die Ebene der beiden ersten mit der Ebene der beiden letzten. Um zwischen diesen beiden Denkmöglichkeiten die Entscheidung zu treffen, genügt es, etwa den gemeinsamen Punkt und damit auch die Ebene der Geraden $X_{(l)}, X_{(r)}$ zu finden. Es ergeben sich dann aber sogleich alle Schnittpunkte und gemeinsamen Ebenen der vier Geraden, und es kann ausfindig gemacht werden, welche von ihnen mit dem Speer X oder Pfeil $\mathfrak{x}_l$, $\mathfrak{x}_r$ vereinigt liegen. Diese stellen wir nunmehr zusammen; zu grösserer Bequemlichkeit fügen wir auch die Coordinaten des Speeres X ausgedrückt als Funktion der Parameter ξ und die durch Auflösung hieraus entstehenden Gleichungen hinzu.

Um einigermaasen lesbare Formeln zu erzielen, schreiben wir an Stelle

von $\xi_{l\kappa}, \xi_{r\kappa}$
einfacher l_κ, r_κ. $(\kappa = 1, 2)$.

Die Coordinaten der zwei Punkte und der zwei Ebenen, die dem Speer oder Pfeil X und ausserdem der absoluten Fläche angehören, unterscheiden wir ebenfalls durch Indices l und r. Inwiefern das berechtigt ist, soll sogleich dargelegt werden.

(54). *Coordinaten des reellen Pfeiles* $\mathfrak{x}_l$, $\mathfrak{x}_r$, *ausgedrückt durch dessen Parameter:*

$$\begin{aligned}
\mathfrak{x}_{l0} &= l_1\bar{l}_1 + l_2\bar{l}_2, & \mathfrak{x}_{r0} &= r_1\bar{r}_1 + r_2\bar{r}_2,\\
\mathfrak{x}_1 &= l_1\bar{l}_2 + l_2\bar{l}_1, & \mathfrak{x}_2 &= r_1\bar{r}_2 + r_2\bar{r}_1,\\
-i\,.\,\mathfrak{x}_3 &= l_1\bar{l}_2 - l_2\bar{l}_1, & -i\,.\,\mathfrak{x}_4 &= r_1\bar{r}_2 - r_2\bar{r}_1,\\
\mathfrak{x}_5 &= l_1\bar{l}_1 - l_2\bar{l}_2, & \mathfrak{x}_6 &= r_1\bar{r}_1 - r_2\bar{r}_2.
\end{aligned}$$

Die zugehörigen *Speercoordinaten* haben diese Werthe (oder sind zu ihnen proportional):

$$X_0 = \mathfrak{x}_{l0} \cdot \mathfrak{x}_{r0},$$
$$X_1 = \mathfrak{x}_{r0} \cdot \mathfrak{x}_1, \quad X_3 = \mathfrak{x}_{r0} \cdot \mathfrak{x}_3, \quad X_5 = \mathfrak{x}_{r0} \cdot \mathfrak{x}_5,$$
$$X_2 = \mathfrak{x}_{l0} \cdot \mathfrak{x}_2, \quad X_4 = \mathfrak{x}_{l0} \cdot \mathfrak{x}_4, \quad X_6 = \mathfrak{x}_{l0} \cdot \mathfrak{x}_6.$$

(55). *Parameter eines reellen Pfeils ausgedrückt durch die Pfeilcoordinaten:*

$$l_1 : l_2 = \mathfrak{x}_{l0} + \mathfrak{x}_5 : \mathfrak{x}_1 + i\mathfrak{x}_3 = \mathfrak{x}_1 - i\mathfrak{x}_3 : \mathfrak{x}_{l0} - \mathfrak{x}_5,$$
$$r_1 : r_2 = \mathfrak{x}_{r0} + \mathfrak{x}_6 : \mathfrak{x}_2 + i\mathfrak{x}_4 = \mathfrak{x}_2 - i\mathfrak{x}_4 : \mathfrak{x}_{r0} - \mathfrak{x}_6.$$

(56). *Ebenen der absoluten Fläche durch den reellen Pfeil* (l, r):

$$\left.\begin{aligned} u_{l0} &= i\{l_1 r_2 - l_2 r_1\}, \\ u_{l1} &= \{l_1 r_1 - l_2 r_2\}, \\ u_{l2} &= i\{l_1 r_1 + l_2 r_2\}, \\ u_{l3} &= -\{l_1 r_2 + l_2 r_1\}. \end{aligned}\right\} \begin{array}{c}\text{Linkseitige} \\ \text{Ebene;} \\ \text{Erzeugende} \\ X_{(l)},\ X_{(r)}.\end{array}$$

$$\left.\begin{aligned} u_{r0} &= -i\{\bar l_1 \bar r_2 - \bar l_2 \bar r_1\}, \\ u_{r1} &= \{\bar l_1 \bar r_1 - \bar l_2 \bar r_2\}, \\ u_{r2} &= -i\{\bar l_1 \bar r_1 + \bar l_2 \bar r_2\}, \\ u_{r3} &= -\{\bar l_1 \bar r_2 + \bar l_2 \bar r_1\}. \end{aligned}\right\} \begin{array}{c}\text{Rechtseitige} \\ \text{Ebene;} \\ \text{Erzeugende} \\ \bar X_{(l)},\ \bar X_{(r)}.\end{array}$$

(57). *Punkte der absoluten Fläche auf dem reellen Pfeil* (l, r):

$$\left.\begin{aligned} x_{l0} &= -i\{l_1 \bar r_1 + l_2 \bar r_2\}, \\ x_{l1} &= \{l_1 \bar r_2 + l_2 \bar r_1\}, \\ x_{l2} &= i\{l_1 \bar r_2 - l_2 \bar r_1\}, \\ x_{l3} &= \{l_1 \bar r_1 - l_2 \bar r_2\}. \end{aligned}\right\} \begin{array}{c}\text{Linkseitiger} \\ \text{Punkt;} \\ \text{Erzeugende} \\ X_{(l)},\ \bar X_{(r)}.\end{array}$$

$$\left.\begin{aligned} x_{r0} &= i\{\bar l_1 r_1 + \bar l_2 r_2\}, \\ x_{r1} &= \{\bar l_1 r_2 + \bar l_2 r_1\}, \\ x_{r2} &= -i\{\bar l_1 r_2 - \bar l_2 r_1\}, \\ x_{r3} &= \{\bar l_1 r_1 - \bar l_2 r_2\}, \end{aligned}\right\} \begin{array}{c}\text{Rechtseitiger} \\ \text{Punkt;} \\ \text{Erzeugende} \\ \bar X_{(l)},\ X_{(r)}.\end{array}$$

Es folgt hieraus zum Beispiel:

$$x_{r0}\,x_{l1} - x_{l0}\,x_{r1} + x_{r2}\,x_{l3} - x_{l2}\,x_{r3} = i\mathfrak{x}_{r0} \cdot \mathfrak{x}_1,$$
$$x_{r0}\,x_{l1} - x_{l0}\,x_{r1} - x_{r2}\,x_{l3} + x_{l2}\,x_{r3} = i\mathfrak{x}_{l0} \cdot \mathfrak{x}_2;$$
$$u_{r2}\,u_{l3} - u_{l2}\,u_{r3} + u_{r0}\,u_{l1} - u_{l0}\,u_{r1} = i\mathfrak{x}_{r0} \cdot \mathfrak{x}_1,$$
$$u_{r2}\,u_{l3} - u_{l2}\,u_{r3} - u_{r0}\,u_{l1} + u_{l0}\,u_{r1} = i\mathfrak{x}_{l0} \cdot \mathfrak{x}_2.$$

Die Formeln (55)-(57) zeigen nunmehr, dass man nach Adjunction der Irrationalität $\sqrt{-1}$ die Punkte und Ebenen, die einem reellen Speer oder Pfeil und zugleich der absoluten Fläche angehören, rational trennen kann. Ferner ist deutlich, dass die gegenseitige Beziehung dieser Figuren durch reelle Bewegungen nicht zerstört werden kann, dass es unmöglich ist, z. B. die Figuren X und u_r gleichzeitig mit den Figuren X und u_l zur Deckung zu bringen. Daraus allein aber ergiebt sich natürlich noch nicht, dass es einen Sinn hat, die beiden Ebenen u_r und u_l durch die Epitheta rechts und links unterscheiden zu wollen, und ihnen diese Epitheta gerade in dieser Reihenfolge anzuheften.

Wir betrachten nun eine Ebene u, die sich um den Speer X im positiven Sinne drehen möge. ϕ sei der Winkel zwischen dieser Ebene und einer bestimmten Ebene durch X, so dass wachsenden Werthen von ϕ der Umlauf im positiven Sinne entspricht. Setzen wir dann die Veränderliche ϕ ins complexe Gebiet fort, so erhalten wir das ganze Bündel der reellen und imaginären Ebenen durch X, mit Ausnahme der beiden, die der absoluten Fläche angehören. Diese beiden Ebenen aber entstehen durch je einen Grenzübergang. Die eine wird erhalten, wenn man in $\phi = \psi + i\chi$ die reelle Grösse χ über alle Grenzen wachsen, die andere, wenn man sie unbegrenzt abnehmen lässt ($\psi =$ const., $\chi = + \infty$, $\chi = - \infty$). Deutet man ψ und χ in üblicher Weise in der *Gauss'schen* Ebene (ψ-Axe nach rechts, χ-Axe nach oben) so geht man im ersten Falle nach *links* von der im positiven Sinne durchlaufenen ψ-Axe, im zweiten nach *rechts*. Deutet man andrerseits, wiederum in der *Gauss'schen* Ebene, die Veränderlichen ψ und χ so, dass $\chi = 0$ ist auf dem Einheitskreis, und ψ dessen Bogen bedeutet, bei positiver Umlaufung des Kreises (entgegengesetzt dem Sinne des Uhrzeigers) so werden dadurch den Grenzwerthen $\chi = + \infty$, $\chi = - \infty$ der Reihe nach die Punkte 0 und ∞ der Zahlenebene zugeordnet: diese liegen aber wieder *links* und *rechts* von dem im positiven Sinne durchlaufenen Orte $\chi = 0$. Von der letzten Deutung endlich kann man zur Zahlenkugel übergehen. Diese werden wir in *gleicher* Weise orientiren, wie unsere beiden Bildkugeln (was zwar keine Nothwendigkeit, aber das Nächstliegende ist). Sie geht dann in die Zahlenebene, deren Einheitskreis den Aequator der Zahlenkugel ausschneiden möge, über durch stereographische Projection aus dem *Südpol.* Die Bilder der Punkte 0 und ∞ sind nun auf der Kugel Nordpol und Südpol, da der positive Drehungssinn auf dem Aequator der von West nach Ost sein soll. (S. 127.) Wieder liegen also die Bilder der beiden ausgezeichneten Ebenen in Bezug auf den orientirten Aequator *links* und *rechts.*

Wir wissen also nun, dass es einen Sinn hat, die beiden Ebenen ($\chi = + \infty$) und $\chi = - \infty$), mit den Beiwörtern links und rechts in Verbindung zu setzen, und es bleibt nur noch festzustellen, welche der unter (56) bezeichneten Ebenen dem Grenzübergang ($\chi = +\infty$) und welche dem Grenzübergang ($\chi = -\infty$) entspricht. Dies kann an irgend einem Zahlenbeispiel ausgeführt werden, z. B. an diesem: $l_1 = l_2 = r_1 = r_2 = 1$. Es ergiebt sich dann die Zuordnung von u_l und ($\chi = +\infty$), u_r und ($\chi = -\infty$).

Ebenso lassen sich die bei den Formeln (57) angebrachten Bezeichnungen rechtfertigen.

Wir fassen nun das Wesentlichste dieser Ueberlegungen zusammen:

Die Bestimmung der Ebenen und Punkte der absoluten Fläche, die einem reellen Speer oder Pfeil angehören, verlangt nur die Adjunction einer numerischen Irrationalität, nämlich der imaginären Einheit.[*]

Die reellen Speere oder Pfeile X und z. B. die imaginären Tangentialebenen u der absoluten Fläche lassen sich (auf zwei Arten) in eine wechselweise EINDEUTIGE *Beziehung setzen, derart, dass der Speer X in der Ebene u liegt, also deren reelle Gerade überdeckt.*

Die beiden Tangentialebenen durch X können durch die Epitheta links und rechts unterschieden werden.

Die linkseitige Ebene u_l z. B. ist dadurch bestimmt, dass sie in dem Gauss'schen oder Riemann'schen Bilde des (zweidimensionalen) Büschels aller reellen und imaginären Ebenen durch X LINKS *liegt, wenn man die reellen Ebenen dieses Büschels im positiven Sinne durchläuft.*

Eine der genannten Zuordnungen wird dann durch die Festsetzung bewirkt, dass dem Pfeil X stets die ihn enthaltende LINKSEITIGE *Tangentialebene u_l der absoluten Fläche entsprechen soll, und umgekehrt.*

Die conjugirt-imaginäre und also rechtseitige Ebene u_r entspricht dann dem umgekehrten Pfeil, in Bezug auf den sie linkseitige Tangentialebene ist.

Man bestätigt das durch leichte Rechnung mit Hülfe der Formeln (55), (56).

Schliesslich ergiebt sich noch eine weitere wichtige Folgerung.

Jede Tangentialebene u_l der absoluten Fläche enthält zwei Erzeugende $X_{(l)}$ und $X_{(r)}$ dieser Fläche, und diese haben auf den beiden Zahlenkugeln genau dieselben Bilder ξ_l und ξ_r oder X_l und X_r wie der reelle Speer oder Pfeil X, der zu u_l gehört, nach der früher ausgeführten Zuordnung. Unterwirft man nun die Ebene u_l einer beliebigen Bewegung mit *complexen* Parametern, so werden die Erzeugenden $X_{(l)}$, $X_{(r)}$ in allgemeinster Weise linear transformirt. Die

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[*] Ist die absolute Fläche nicht in der von uns vorausgesetzten Form dargestellt, sondern durch eine nicht-specialisirte quadratische Form gegeben, so tritt an Stelle dieser Grösse die Quadratwurzel aus der negativ genommenen Discriminante dieser Form.

Parameter $\xi_{l1} : \xi_{l2}$ und $\xi_{r1} : \xi_{r2}$ unterliegen jetzt der allgemeinsten linearen Transformation. Die Bildkugeln werden eigentlich-collinear transformirt, und die Speere oder Pfeile X werden dadurch einer eigentlichen cyclischen Transformation unterworfen (S. 143).

Die eigentlichen cyclischen Transformationen der reellen (Speere oder) Pfeile können also auch in folgender Weise erklärt werden:

Jede Tangentialebene der absoluten Fläche hat nach Obigem ein eindeutig bestimmtes Bild in dem ihr zugeordneten Pfeil. Unterwirft man nun die Tangentialebenen den ∞^{12} reellen und complexen Bewegungen des elliptischen Raumes, so werden die zugehörigen Pfeile entsprechend vertauscht, und zwar durch die ∞^{12} reellen eigentlichen cyclischen Transformationen.

Die 2. ∞^{6} reellen und complexen Schiebungen liefern auf diese Weise die beiden Arten cyclischer Schiebungen im reellen Pfeilcontinuum. (Vgl. S. 143).

Bis hierher haben wir den gelegentlich verwendeten Begriff „complexer Pfeil“ nur durch ein System von zweimal vier oder viermal zwei homogenen Coordinaten, also *formal* erklärt, während wir für den verwandten Begriff des complexen Speeres eine Zurückführung auf bekannte Begriffe der (complexen) Nicht-Euklidischen Geometrie zur Verfügung hatten. Nunmehr ergiebt sich eine entsprechende Deutung auch für den Pfeil.

Setzen wir nämlich in die Formeln (54) - (57) statt der conjugirt-complexen Verhältnissgrössen $l_1 : l_2$ und $\bar{l}_1 : \bar{l}_2$, $r_1 : r_2$ und $\bar{r}_1 : \bar{r}_2$ beliebige Verhältnissgrössen ein, so erhalten wir statt der conjugirt-imaginären Ebenen u_l, u_r und Punkte x_l, x_r irgend welche in bestimmte Reihenfolge gesetzte Ebenen und Punkte der absoluten Fläche, die in der angezeigten Weise durch ein auf dieser Fläche verlaufendes Vierseit $X_{(l)}$, $X_{(r)}$, $\bar{X}_{(l)}$, $\bar{X}_{(r)}$ von nunmehr ganz beliebigen Erzeugenden verbunden sind. Zwischen dieser Figur und den Pfeilcoordinaten besteht eine eindeutig-umkehrbare Beziehung. Wenn wir wollen, können wir daher sagen: Diese Figur *ist* der Pfeil.

Da die Anordnung der beiden Punkte x_l, x_r auf dem Pfeil der Anordnung der übrigen Figuren u_l, u_r und $X_{(l)}$, $X_{(r)}$, $\bar{X}_{(l)}$, $\bar{X}_{(r)}$ vollkommen bestimmt, so wollen wir — mit Rücksicht auf die in einer späteren Arbeit zu behandelnde hyperbolische Geometrie — nur von diesen Punkten reden. Auch scheint es uns zweckmässig, die Punkte x_l, x_r statt durch die Worte links und rechts, die sich auf den Fall *reeller* Pfeile des *elliptischen* Raumes beziehen, allgemein durch die Worte *Anfang* und *Ende des Pfeiles* (*oder Speeres*) zu unterscheiden. Dann können wir also sagen:

Ein (*reeller oder complexer*) SPEER *kann in der Regel aufgefasst werden als eine gerade Linie, deren beide Schnittpunkte mit der absoluten Fläche bekannt und als Anfang x_l und Ende x_r des Speeres in eine bestimmte Reihenfolge gebracht sind.*

Fallen jedoch diese Schnittpunkte zusammen, oder sind sie unbestimmt, so fällt der Begriff des Speeres zusammen mit dem Begriff einer Geraden (*nämlich einer Minimalgeraden oder insbesondere einer Erzeugenden der absoluten Fläche.*)

Ein (*reeller oder complexer*) PFEIL *dagegen* IST *ein Paar von Punkten der absoluten Fläche, die als Anfang x_l und Ende x_r des Pfeiles bestimmt geordnet sind.*

Der Begriff des Pfeiles fällt also niemals zusammen mit dem einer geraden Linie.

Es ergiebt sich nunmehr die gegenseitige Beziehung der Begriffe Speer und Pfeil, die wir in folgender Tafel darstellen:

∞^8 *Speere „allgemeiner Lage,“* darunter die ∞^4 reellen Speere des elliptischen Raumes.	∞^8 *Pfeile „allgemeiner Lage,“* darunter die ∞^4 reellen Pfeile des elliptischen Raumes.

Je *zwei* solche Speere oder Pfeile gehören zur selben Geraden, die *nicht* Minimalgerade, nicht Tangente der absoluten Fläche ist.

∞^6 *„Minimalspeere allgemeiner Lage.“* Zu jeder Tangente, die *nicht* Erzeugende ist, gehört *ein* solcher Speer.	∞^4 *„Punktpfeile,“* das sind Pfeile, deren jeder denselben Anfangs- und Endpunkt $x_l = x_r$ hat. Sie entsprechen auch den Speeren der nächsten Classe.
2 . ∞^2 *Erzeugende der absoluten Fläche,* deren jede ein specieller Minimalspeer ist. Die beiden letzten Classen bilden zusammen den Complex aller Minimalspeere.	2 . ∞^6 *„accessorische Pfeile“*, nämlich Punktepaare x_l, x_r die durch (mindestens) je eine Erzeugende der absoluten Fläche verbunden werden können, darunter die zuvor genannten Punktpfeile.

Im Pfeilcontinuum gibt es also, an Stelle des einen *absoluten Complexes,* der von den Minimalspeeren oder Minimalgeraden gebildet wird, *zwei accessorische Complexe,* die (wie die Erzeugenden der absoluten Fläche) durch die Epitheta links und rechts zu unterscheiden sind. Diese beiden Complexe durchdringen einander in der *absoluten Congruenz,* die von allen Punktpfeilen gebildet wird, und in der Speermannigfaltigkeit kein Analogon hat.

Alle diese Figuren bleiben in Ruhe bei den ∞^{12} complexen Bewegungen.

Das achtdimensionale Continuum aller complexen Pfeile hat, im Sinne der Analysis Situs, eben denselben Zusammenhang, wie das Continuum aller Quadrupel reeller Punkte, die man einzeln vier reelen Kugelflächen entnehmen kann.

Es ist selbstverständlich, dass es algebraische Punktmannigfaltigkeiten giebt, die *vollkommen* eindeutig, umkehrbar und stetig auf das Speercontinuum abgebildet werden können, und andere, die ebenso auf das Pfeilcontinuum abgebildet werden können. Wir werden (in der folgenden Abhandlung) solche Mannigfaltigkeiten nachweisen, die noch weitere bemerkenswerthe Eigenschaften haben.

Natürlich kann man die Reihenfolge unserer Betrachtungen umkehren. Man kann mit der Parameterdarstellung (51) der Erzeugenden der absoluten Fläche beginnen. Von dieser kommt man dann sogleich zu dem Begriffe des Pfeils. Die Gruppe aller eigentlichen cyclischen Transformationen der Pfeile wird dann wie folgt zu erklären sein:

Eigentliche cyclische Transformationen der Pfeile sind solche, die Anfangs- und Endpunkt eines jeden Pfeiles dem Anfangs- und Endpunkt des entsprechenden Pfeiles durch irgend zwei (complexe) Bewegungen zuordnen.

Die so erklärte nach der üblichen Terminologie zwölfgliedrige, in Wirklichkeit vierundzwanziggliedrige Gruppe hat vier paarweise vertauschbare sechsgliedrige invariante Untergruppen, und daher im Ganzen vierzehn invariante continuirliche oder sogenannte gemischte Untergruppen, von denen aber hier, wo die reellen Figuren im Mittelpunkt der Betrachtung stehen, nur zwei zur Sprache gekommen sind. Es sind dies die folgenden:

1) und 2). Die beiden Gruppen mit je zwölf reellen oder sechs complexen wesentlichen Parametern, die links- oder rechts- syntaktische Pfeile in ebensolche überführen: *Die beiden Gruppen cyclischen Schiebungen.*

Ausser diesen kommen noch besonders in Betracht gewisse nicht- invariante Untergruppen, nämlich:

3). *Die Gruppe der Bewegungen* selbst, die von zwölf reellen oder sechs complexen wesentlichen Parametern abhängt.

4). Die Gruppe der ∞^{12} cyclischen Transformationen, bei denen Anfangs- und Endpunkt eines jeden Pfeiles *conjugirt-complexen* Bewegungen unterworfen werden: Die Gruppe der *reellen* eigentlichen cyclischen Transformationen.

Man sieht, dass ein solcher Entwickelungsgang, der natürlich ebenfalls zu den Speeren und ihren sphärischen Bildern führt, seine systematischen und vielleicht auch pädagogischen Vorzüge haben wird. Indessen schien es uns doch angezeigt, die Einführung imaginärer Figuren so lange als möglich zu vermeiden.

Das nächste Ziel dieser einleitenden Abhandlung glauben wir nunmehr erreicht zu haben.

Eine besondere wichtige Folgerung, die sich unmittelbar ergiebt, ist die, dass die cyclischen Transformationen der Pfeile (genauer eine zu dieser Gruppe ähnliche Gruppe, und übrigens noch viel umfassendere vorläufig unberührt gelassene Gruppen von Transformationen) auch aus der Geometrie des *hyperbolischen* Raumes abgeleitet werden können. Dieser Gegenstand soll, mit Anwendungen auf Differentialgeometrie, in einer folgenden Abhandlung dargelegt werden.

Eine andere Erweiterung der vorgetragenen Theorie wollen wir wenigstens noch anhangsweise erwähnen.

Der Verfasser bezeichnet mit dem Worte *Soma* die Lage eines starren Körpers. Es kann nun das Soma im elliptischen Raume durch Coordinaten dargestellt werden, die völlig analog sind denen eines *Linienkreuzes* (nämlich durch die doppelt-homogenen reellen Parameter a_ι und b_ι), und ebenso das Soma im sphärischen Raume durch Coordinaten, die analog sind denen einer *Geraden* (dieselben Parameter, verbunden durch die quadratische Gleichung $N(a) = N(b)$). Endlich können die Somen im hyperbolischen Raume durch ebensolche Coordinaten dargestellt werden, wie die complexen Punkte eines projectiven Continuums, abgesehen von einer gewissen Ungleichung, durch die die ∞^4 Punkte einer Fläche zweiter Ordnung ausgeschlossen werden.

Das Soma aber ist der Grundbegriff der Kinematik.

Die Nicht-Euklidische Kinematik kann also in allen Fällen aus Begriffen entwickelt werden, die nur durch die Dimensionenzahlen sich unterscheiden von solchen, mit denen wir es in der vorliegenden Abhandlung zu thun hatten.

Wie das geschehen kann, darüber findet der Leser Andeutungen in des Verfassers Geometrie der Dynamen (§40, und Anhang, S. 535-594), wo der (ziemlich viel verwickeltere) Fall der Euklidischen Kinematik nach solchen Grundsätzen behandelt worden ist.

III.

Schraubenflächen als Extreme.

In der folgenden Mittheilung sollen in aller Kürze einige Sätze über Schraubenflächen abgeleitet werden, die eine einfache Anwendung der in der vorhergehenden Abhandlung II entwickelten Theorie darstellen. Einer dieser Sätze besagt, dass man der üblichen Abbildung der Geraden des Plücker'schen Liniencontinuums auf die Punkte einer quadratischen Mannigfaltigkeit eine Beziehung zur Nicht-Euklidischen Geometrie geben kann:

Das Continuum der reellen Geraden im elliptischen (oder sphärischen) Raume lässt sich auf eine reelle quadratische Mannigfaltigkeit, die im elliptischen Raume von fünf Dimensionen verläuft, derart abbilden, dass den Schraubenflächen die geodätischen Linien auf dieser Mannigfaltigkeit zugeordnet werden.

Eine solche Abbildung erhalten wir auf eine sehr einfache Weise. Wir deuten erstens die homogenen Liniencoordinaten $X_1 \dots X_6$ als homogene Punktcoordinaten im Raume R_5. Aus den Bewegungen im dreidimensionalen elliptischen Raume geht dann eine projective Gruppe G_6 im Raume R_5 hervor. Die Gleichung der quadratischen Mannigfaltigkeit, die wir mit M_4^2 bezeichnen wollen, wird:

$$X_1^2 + X_3^2 + X_5^2 = X_2^2 + X_4^2 + X_6^2 \tag{1}$$

Zweitens wählen wir unter den quadratischen Mannigfaltigkeiten des Raumes R_5, die elliptische Maassbestimmungen definiren (und also keine reellen Punkte haben), irgend eine aus, die bei der Gruppe G_6 in Ruhe bleibt. Alle diese lassen sich leicht bestimmen. Sie haben Gleichungen der Form

$$c_l(X_1^2 + X_3^2 + X_5^2) + c_r(X_2^2 + X_4^2 + X_6^2) = 0, \tag{2}$$

worin die Constanten c_l und c_r von Null verschiedene *positive* Verhältnissgrössen sind.

Benutzen wir also jetzt die Mannigfaltigkeit (2) als absolutes Gebilde einer Maassbestimmung vom Krümmungsmaasse Eins, und bedienen wir uns der in der vorausgehenden Abhandlung II eingeführten Bezeichnungen, so erhalten wir für die geradlinig gemessene Entfernung $[Y, Z]$ zweier auf M_4^2 gelegener Punkte Y, Z die Gleichung

$$(c_l + c_r)\sin^2 \tfrac{1}{2}\,[Y, Z] = c_l \,.\, \sin^2 \tfrac{1}{2}\,(Y, Z)_l + c_r \,.\, \sin^2 \tfrac{1}{2}\,(Y, Z)_r. \tag{3}$$

Für entsprechende Bogenelemente dS und ds_l, ds_r ergiebt sich heraus

$$(c_l + c_r)\, dS^2 = c_l\, ds_l^2 + c_r\, ds_r^2. \tag{4}$$

Lassen wir also die beiden Punkte X_l und X_r auf den zugehörigen Bildkugeln mit constanten Geschwindigkeiten $2\Theta_l'$ und $2\Theta_r'$, deren mindestens eine nicht verschwindet, in geodätischen Linien (also auf grössten Kreisen) sich bewegen, so bewegt sich der Punkt X auf M_4^2 ebensfalls in einer geodätischen Linie mit constanter Geschwindigkeit $2\Theta'$, die man aus der Gleichung

$$(c_l + c_r)\,.\,\Theta'^2 = c_l\,.\,\Theta_l'^2 + c_r\,.\,\Theta_r'^2 \tag{5}$$

erhält. Zugleich ergiebt sich die Länge $\{Y, Z\}$ irgend eines geodätischen Bogens zwischen zwei Punkten Y, Z von M_4^2:

$$(c_l + c_r)\,.\,\{Y, Z\}^2 = c_l\,.\,(Y, Z)_l^2 + c_r (Y, Z)_r^2. \tag{6}$$

Andrerseits durchläuft, im gewöhnlichen elliptischen Raume R_3, der Speer X eine Schraubenfläche mit constanter Drehungsgeschwindigkeit $2\vartheta'$ und Gleitungsgeschwindigkeit $2\eta'$:

$$2\vartheta' = \Theta_l' + \Theta_r', \quad 2\eta' = \Theta_l' + \Theta_r'. \tag{7}$$

Hiermit ist unsere Behauptung erwiesen.

Es ist leicht, den Verlauf der geodätischen Linien auf M_4^2 zu übersehen, und sich überhaupt die Einzelheiten der benutzten Abbildung deutlich zu machen.

Die Bilder der reellen parataktischen Congruenzen sind zwei Schaaren von je ∞^2 Kugeln. Alle Kugeln derselben Schaar sind zu einander congruent (und im Falle $c_l = c_r$ auch die verschiedener Schaaren). Zwei verschiedene Kugeln derselben Schaar schneiden einander nicht, zwei Kugeln verschiedener Schaaren aber schneiden sich immer, und zwar unter constantem (im Falle $c_l = c_r$ unter rechtem) Winkel in zwei Punkten, die Bilder der Geraden eines Linienkreuzes sind, und auf beiden Kugeln einander diametral gegenüber liegen. Durch jeden Punkt von M_4^2 gehen zwei solche Kugeln mit Durchmessern, deren Längen einander zu π ergänzen.

Wenn nun zwei Punkte Y, Z *nicht* auf einer solchen Kugel liegen, so haben sie in Bezug auf die sie verbindenden geodätischen Linien allgemeines Verhalten. Das heisst, es giebt zwischen ihnen keine Continua von gleichlangen geodätischen Wegen. Insbesondere tritt das auch ein für die ∞^7 Punktepaare, die geodätisch durch *gerade* Linien verbunden werden können.

Solche Punkte Y, Z dagegen, die durch *eine* der genannten Kugeln verbunden sind, und also auf dieser einander nicht diametral gegenüberliegen, verhalten sich, wie wir etwa sagen können, *im ersten Grade singulär* in Bezug auf die geodätischen Linien. Es giebt zwischen ihnen eine isolirte geodätische Linie, die ein grösster Kreis auf der zugehörigen Kugel ist. Auf ihr liegt der absolut-kürzeste Weg zwischen Y und Z. Ausserdem aber giebt es noch unendlich viele Continua von je ∞^1 gleichlangen geodätischen Wegen zwischen je zwei solchen Punkten. Punktepaare der Art sind ∞^6 vorhanden. *Im zweiten Grade singulär* verhalten sich dann die ∞^4 gepaarten Punkte. Solche Punkte Y, Z werden erstens durch zwei Schaaren von je ∞^1 geodätischen Wegen verbunden, die auf je einer der zwei Kugeln verlaufen, und die constanten Längen

$$\sqrt{\frac{c_l}{c_l + c_r}} \cdot \pi, \qquad \sqrt{\frac{c_r}{c_l + c_r}} \cdot \pi$$

haben. Unter diesen Wegen befinden sich die absolut kürzesten Wege zwischen Y, Z. Ausserdem aber giebt es noch unendlich viele Schaaren von je ∞^2 gleichlangen geodätischen Wegen zwischen Y und Z.

Das angewendete Verfahren liefert noch mehrere ähnliche Sätze. Ja, es liefert deren unbegrenzt viele, und es wird keine sonderliche Schwierigkeit haben, sie alle zu ermitteln. Wir wollen uns jedoch mit einigen charakteristischen Beispielen begnügen, um diese Art von Ueberlegungen nicht über Gebühr auszudehnen.

Wir betrachten jetzt als Raumelement den reellen Speer. Wir gelangen dann sofort zu einer Bildmannigfaltigkeit *vierter* Ordnung, M_4^4, die in einem projectiven Punktcontinuum von *sechs* Dimensionen verläuft:

$$X_0^2 = X_1^2 + X_3^2 + X_5^2 = X_2^2 + X_4^2 + X_6^2. \tag{8}$$

Man sieht, dass M_4^4 auf M_4^2 zwei-eindeutig bezogen ist, was weiterhin die Folge haben wird, dass die Punktepaare Y, Z auf M_4^4, denen in Bezug auf die noch zu erklärenden geodätischen Linien eine Sonderstellung zukommt, ein verwickelteres Verhalten darbieten als in dem eben betrachteten Falle.

Auch M_4^4 gestattet nun, wie M_4^2, eine zu den elliptischen Bewegungen isomorphe continuirliche Collineationsgruppe; und überdies erweist sich diese

Gruppe auch — abweichend von dem Falle der M_4^2 — als durch die Mannigfaltigkeit M_4^4 selbst schon bestimmt.

Invariante quadratische Mannigfaltigkeiten, die zu elliptischen Maassbestimmungen führen, sind jetzt diese:

$$c_0' X_0^2 + c_l' (X_1^2 + X_3^2 + X_5^2) + c_r' (X_2^2 + X_4^2 + X_6^2) = 0 \qquad (9)$$

wofern $c_0' > 0$, $c_l' > 0$, $c_r' > 0$.

An Stelle der Gleichung (3) tritt jetzt die Gleichung

$$(c_0' + c_l' + c_r') \sin^2 \tfrac{1}{2} \, [Y, Z] = c_l' . \sin^2 \tfrac{1}{2} (Y, Z)_l + c_r' . \sin^2 \tfrac{1}{2} (Y, Z)_r, \qquad (10)$$

und hieraus ergeben sich ähnliche Folgerungen wie zuvor. Aber die geodätischen Linien auf M_4^4 sind hier Bilder von Schraubenflächen, die als Oerter von *Speeren* aufgefasst werden (S. 140). Auf eine Erörterung der verschiedenartigen Punktepaare Y, Z auf M_4^4 darf wohl verzichtet werden. Doch wollen wir bemerken, dass als Bilder der syntaktischen Congruenzen nunmehr 2. ∞^2 Kugeln erscheinen, von denen zwei verschiedenartige einander in *einem* Punkte schneiden.

Leicht sieht man, dass man in diesem Falle statt einer elliptischen auch eine gewöhnliche Euklidische Maassbestimmung benutzen kann. Man deute zum Beispiel die Grössen

$$\sqrt{\frac{c_l}{c_l + c_r}} \cdot \frac{X_1}{X_0}, \quad \sqrt{\frac{c_r}{c_l + c_r}} \cdot \frac{X_2}{X_0},$$

u. s. w. als rechtwinkliche Cartesische Coordinaten. Man kommt dann wieder zu den Gleichungen (4) . . . (7). Aber die Bedeutung dieser Gleichungen ist nun eine andere, da ein Zeichenwechsel der sämmtlichen Coordinaten einen von dem Punkte X verschiedenen Punkt der Mannigfaltigkeit M_4^4 liefert. Durch Projection aus dem Anfangspunkt der Coordinaten kommt man zu dem zuerst behandelten Beispiel zurück.*)

Wir betrachten jetzt als Raumelement den reellen Pfeil. Aus Pfeilcoordinaten erster Art bilden wir die folgenden Producte:

$$\begin{array}{llll}
\mathfrak{X}_{l0}\,\mathfrak{X}_{r0}, & \mathfrak{X}_{l0}\,\mathfrak{X}_2, & \mathfrak{X}_{l0}\,\mathfrak{X}_4, & \mathfrak{X}_{l0}\,\mathfrak{X}_6, \\
\mathfrak{X}_1\,\mathfrak{X}_{r0}, & \mathfrak{X}_1\,\mathfrak{X}_2, & \mathfrak{X}_1\,\mathfrak{X}_4, & \mathfrak{X}_1\,\mathfrak{X}_6, \\
\mathfrak{X}_3\,\mathfrak{X}_{r0}, & \mathfrak{X}_3\,\mathfrak{X}_2, & \mathfrak{X}_3\,\mathfrak{X}_4, & \mathfrak{X}_3\,\mathfrak{X}_6, \\
\mathfrak{X}_5\,\mathfrak{X}_{r0}, & \mathfrak{X}_5\,\mathfrak{X}_2, & \mathfrak{X}_5\,\mathfrak{X}_4, & \mathfrak{X}_5\,\mathfrak{X}_6.
\end{array}$$

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*) Vergleiche die Anmerkung auf Seite 121.

Diese deuten wir als homogene Punktcoordinaten $\Xi_{i\varkappa}$ $(i = 0, 1, 3, 5,$ $\varkappa = 0, 2, 4, 6)$ in einem projectiven Continuum R_{15} von 15 Dimensionen. Dann erhalten wir eine Bildmannigfaltigkeit der Pfeile, deren Ordnung mit Hülfe einer im vorliegenden Falle einwandsfreien Continuitätsbetrachtung = 24 gefunden wird. Diese Mannigfaltigkeit M_4^{24} gestattet und bestimmt eine (sogenannte halb-einfache) reelle continuirliche zwölf-gliedrige Collineationsgruppe*), das Bild der eigentlichen cyclischen Transformationen der Pfeile. Die Bewegungen im elliptischen Raume aber bestimmen eine sechsgliedrige Untergruppe dieser Gruppe, und eine Schaar von ∞^8 invarianten quadratischen Mannigfaltigkeiten ohne reelle Punkte,

$$\begin{gathered}\gamma_0\Xi_{00}^2 + \gamma_l(\Xi_{10}^2 + \Xi_{30}^2 + \Xi_{50}^2) + \\ + \gamma_r(\Xi_{02}^2 + \Xi_{04}^2 + \Xi_{06}^2) + \gamma_{lr}\sum_{i,\varkappa}\Xi_{i\varkappa}^2 = 0 \\ \{\gamma_0, \gamma_l, \gamma_r, \gamma_{lr} > 0,\ i = 1, 3, 5,\ \varkappa = 2, 4, 6\}.\end{gathered} \tag{11}$$

Wählt man irgend eine von diesen als absolutes Gebilde einer elliptischen Maassbestimmung, vom Krümmungsmaasse Eins, so ergiebt sich für die (geradlinig gemessene) Entfernung zweier Punkte H, Z auf M_4^{24} der Ausdruck:

$$\begin{gathered}(\gamma_0 + \gamma_l + \gamma_r + \gamma_{lr})\cos[H, Z] = \\ = \gamma_0 + \gamma_l\cos(Y, Z)_l + \gamma_r\cos(Y, Z)_r + \gamma_{lr}\cos(Y, Z)_l\cos(Y, Z)_r.\end{gathered} \tag{12}$$

Hieraus folgt für entsprechende Bogenelemente:

$$(\gamma_0 + \gamma_l + \gamma_r + \gamma_{lr})\,dS^2 = (\gamma_l + \gamma_{lr})ds_l^2 + (\gamma_r + \gamma_{lr})\,ds_r^2. \tag{13}$$

Man sieht, es ergeben sich wieder ähnliche Folgerungen wie zuvor. Angemerkt zu werden verdient jedoch:

Die beiden Mannigfaltigkeiten M_4^4 und M_4^{24} sind zwar beide frei von Singularitäten, und sie sind birational, und überdies im reellen Gebiete vollkommen eindeutig und stetig, ausserdem auch geodätisch, auf einander abgebildet. Im complexen Gebiete jedoch ist diese Abbildung mit singulären Stellen behaftet.

Die Einzelheiten ergeben sich aus dem Verhältniss der Begriffe Speer und Pfeil. (S. 157).

Schliesslich kann man die *Schraubenflächen* auch als *Oerter von Linienkreuzen* ansehen (S. 140). Deutet man dann die Producte $\mathfrak{x}_i\,\mathfrak{x}_\varkappa$ $\{i = 1, 3, 5, \varkappa = 2, 4, 6\}$ als homogene Punktcoordinaten $\Xi_{i\varkappa}$, so kommt man zu einer Punktmannigfaltigkeit M_4^6, von der sechsten Ordnung, die von zwei Schaaren von je ∞^3 Ebenen beschrieben werden kann — den Bildern der beiden Schaaren parataktischer Congruenzen, die man aus Linienkreuzen bilden kann. Diese Mannigfaltigkeit gestattet und bestimmt eine (halb-einfache) continuirliche Collineationsgruppe mit sechzehn Parametern; und in dieser liegt wieder eine zur Gruppe der Bewegungen im elliptischen Raume isomorphe Untergruppe. Die einzige invariante quadratische Mannigfaltigkeit ist jetzt

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*) Im complexen Gebiete gestattet (und bestimmt) M_4^{24} vierundzwanzig continuirliche Schaaren von je ∞^{24} Collineationen, von denen acht Schaaren reelle Transformationen enthalten. (Vgl. S. 143.)

$$\Sigma \Xi_{ik}^2 = 0, \tag{14}$$

und diese führt zu den Gleichungen

$$\cos\,[H,\, Z] = \cos\,(\mathfrak{y},\, \mathfrak{z})_l \cdot \cos\,(\mathfrak{y},\, \mathfrak{z})_r \tag{15}$$

und

$$dS^2 = ds_l^2 + ds_r^2. \tag{16}$$

Das Weitere ergiebt sich von selbst.

Als Ausdruck für die geodätische Entfernung zweier Punkte Y, Z auf irgend einer der betrachteten Bildmannigfaltigkeiten hat sich stets eine Grösse der Form

$$\{Y,\, Z\} = \sqrt{\mu_l \cdot (Y,\, Z)_l^2 + \mu_r \cdot (Y,\, Z)_r^2} \tag{17}$$

ergeben, wo $\mu_l > 0$, $\mu_r > 0$. Wir wollen nun die einfachste Annahme $\mu_l = \mu_r = 1$ machen, was entweder schon von vorn herein zutrifft, oder (in allen Fällen) dadurch bewirkt werden kann, dass man $c_l = c_r$, $c'_l = c'_r$, $\gamma'_l = \gamma'_r$ setzt, und das Krümmungsmaass oder die Längeneinheit des benutzten höheren Raumes auf geeignete Weise abändert. Unter dieser Voraussetzung wollen wir die mehrdeutige Grösse $\{Y, Z\}$ schlechtweg *geodätische Entfernung der reellen Speere oder Pfeile**) Y, Z nennen.

Betrachten wir anderseits irgend eine die Speere oder Pfeile Y, Z verbindende Schraubenfläche. X sei eine Axe dieser Fläche, so dass Y durch gleichförmige Schraubung um X schliesslich in Z übergeht. Irgend ein Punkt von Y beschreibt dann eine orthogonale Trajectorie der Erzeugenden der Schraubenfläche, eine Schraubenlinie (zugleich asymptotische Linie der Schraubenfläche), die zwischen Y und Z eine bestimmte Bogenlänge hat. Unter diesen Schraubenlinien giebt es zwei ausgezeichnete von gleicher Bogenlänge, die nämlich, deren erzeugende Punkte von X und von der Polare von X den Abstand $\equiv \frac{\pi}{4}$ mod. $\frac{\pi}{2}$ haben, also, wie man sagen kann, von beiden Geraden gleichweit abstehen. Diese Schraubenlinien wollen wir etwa *die mittleren Schraubenlinien zwischen Y, Z* nennen.

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*) Oder nach Umständen, der Geraden oder der Linienkreuze Y, Z.

Es gilt nun der einfache Satz:

*Die geodätische Entfernung zweier reeller Speere Y, Z ist bestimmt, nachdem man diese durch eine Schraubenfläche verbunden, und nöthigenfalls über den Weg, auf dem Y in Z übergeführt werden soll, entschieden hat.**) *Sie wird dann gleich dem Doppelten der entsprechenden gemeinsamen Länge der beiden mittleren Schraubenlinien zwischen beiden Speeren.*

In der That, schreitet bei der gleichförmigen Schraubenbewegung, die Y schliesslich in Z überführt, der Schnittpunkt x von Y und X um die Länge $2d\eta$ fort, und der Schnittpunkt x' von Y mit der Polare von X um die Länge $2d\vartheta$, so schreitet, wie eine leichte Rechnung zeigt, irgend ein Punkt auf Y mit der Gleichung

$$\frac{\sqrt{\mu}\,(ux)}{\sqrt{(xx)}} + \frac{\sqrt{\nu}\,(ux')}{\sqrt{(x'x')}} = 0 \tag{18}$$

um eine infinitesimale Strecke fort, deren Längenquadrat gleich ist dem Ausdruck

$$4\,\frac{\mu\,.\,d\eta^2 + \nu\,.\,d\vartheta^2}{\mu + \nu}. \tag{19}$$

Ist $\mu = \nu$, so durchläuft der genannte Punkt eine der beiden mittleren Schraubenlinien, und der letzte Ausdruck reducirt sich auf

$$\tfrac{1}{4}\{(Y,\ Y + dY)_l^2 + (Y,\ Y + dY)_r^2\}. \tag{20}$$

Natürlich kann man auch das Vorzeichen von $\{Y, Z\}$ von dem Sinne abhängig machen, in dem die Schraubenfläche durchlaufen wird.

In der hyperbolischen und in der Euklidischen Geometrie kann man zum Theil entsprechende Entwickelungen vornehmen. Die Schraubenflächen erscheinen auch hier als Lösungen eines Problems der Variationsrechnung, das sich aus der Bildung gewisser quadratischer Differentialformen ergiebt. Sie haben indessen in diesen Fällen *nicht* die Eigenschaft, Extreme zu sein. Auch im elliptischen Raume selbst treten solche Erscheinungen schon auf, wenn man die von uns mit c_l, c_r u. s. w. bezeichneten Constanten verschwinden und in negative Werthe übergehen lässt.

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*) Wenn die Schraubenfläche algebraisch ist, so sind verschiedene Wege möglich.

Dagegen können sich unsere Ueberlegungen vielleicht in anderer Beziehung als nützlich erweisen. Die geodätische Entfernung

$$\{Y, Z\} = \sqrt{(Y, Z)_i^2 + (Y, Z)_r^2} \tag{21}$$

zweier Speere z. B. wird eine ähnliche Behandlung zulassen, wie der gewöhnliche Entfernungsbegriff. Die Geometrie der Speere liefert so ein Beispiel zu der auf Differentialformen gegründeten Mannigfaltigkeitslehre, und zwar zur Theorie der Mannigfaltigkeiten von nicht-constantem Riemann'schem Krümmungsmaass. Die Probleme der gewöhnlichen Flächentheorie können auf diesen Fall übertragen werden und liefern dann neue Gesichtspunkte für die Behandlung der Liniencongruenzen und anderer Figuren.

Schliesslich verdient es wohl noch bemerkt zu werden, dass die zu Ausdrücken der Form (17) gehörige Maassgeometrie auch in einem anderen Gedankenkreise auftritt.

Der Verfasser hat mit Hülfe der Theorie der *Hermite*'schen Formen aus der gewöhnlichen elliptischen und hyperbolischen Geometrie allgemeinere Arten der Maassgeometrie abgeleitet. (Verhandlungen des Heidelberger Mathematiker-Congresses, Leipzig, 1905, S. 312–321 und Math. Ann. Bd. 60, 1905, S. 321). In ähnlicher Weise und zufolge dieser Theorie lässt sich auch die sphärische Geometrie erweitern. Die Geometrie auf einer Kugelfläche insbesondere führt dann zu einer Maassgeometrie in einer vierdimensionalen Mannigfaltigkeit, die ihrem logischen Inhalte nach nicht verschieden ist von der hier besprochenen Maassgeometrie im Continuum der reellen Speere oder Pfeile.

In der auf Seite 159 erwähnten Geometrie der Somen im elliptischen oder sphärishen Raume giebt es ebenfalls Sätze, die den vorgetragenen ähnlich sind.

BONN, *im März* 1906.

Certain Triply Orthogonal Systems of Surfaces.

By L. P. Eisenhart.

In a former paper[*] we have discussed the surfaces which have the same spherical representation of their lines of curvature as pseudospherical surfaces, and for convenience of reference have called them the *A*-surfaces. This paper is devoted to the study of triply orthogonal systems in which all the surfaces in one family are *A*-surfaces; we shall refer to them as *A*-systems.

In §1 we recall the general equations of condition for a triply orthogonal system, first found by Lame, and the formulas which give the expressions for the fundamental functions of the surfaces of a system. These are applied in §2 to the special case of *A*-systems, and it is found that there are three conditions to be satisfied by the spherical representation of the *A*-surfaces of the system, and three others by the coefficients of the linear element of space in terms of these surfaces. From the definition of *A*-surfaces it is seen that given a surface of this kind there is a pseudospherical surface with the same spherical representation of its lines of curvature; we shall say that the latter is the *associated* pseudospherical surface of the former. Given any *A*-system; the spherical representation of the associated pseudospherical surfaces evidently satisfies the three conditions mentioned above, and one finds that the coefficients of the linear element of space in terms of this new system satisfy the other three conditions identically; hence when one has a family of *A*-surfaces satisfying the conditions of Lame, the associated pseudospherical surfaces form a family of Lame. This shows that all *A*-systems can be obtained from pseudospherical systems by the transformation of Combescure.[†] A study of this transformation as applied to *A*-systems is made in §3, and several particular cases are discussed.

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[*] Surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces, American Journal, vol. 27, pp. 113–172.

[†] Bianchi, Lezioni, II, p. 490.

In the previous paper* we established certain transformations for A-surfaces which reduce in the case of pseudospherical surfaces to the well-known transformations of Bianchi and Bäcklund. In §4 we determine the conditions to be satisfied in order that a set of transforms of a Lamé family of A-surfaces shall constitute such a family. It is found that when such a transformation is known for one system it can be applied to all systems with the same spherical representation. It is likewise shown that these transformations admit of a theorem of permutability reducing for pseudospherical systems to the theorem of Bianchi.†

Weingarten was the first to consider a pseudospherical system in which all the pseudospherical surfaces have the same total curvature. The representation upon the sphere takes a form of special interest for this case; it is considered in §5 and all A-systems with this representation are referred to as *Weingarten A-systems.* The moulure surfaces are A-surfaces, and §6 deals with systems in which all the surfaces in one system are of this kind; it is found that in such systems a second family also is composed of moulure surfaces.

In §7 we consider the A-systems for which the orthogonal trajectories of the family of A-surfaces are plane curves. It is shown that so far as the spherical representation is concerned this problem is reducible to the systems with circular trajectories; a large class of the latter are treated at length. One passes then to the consideration of A-systems in which one system of the lines of curvature on the A-surface are plane curves; this occupies §8. §9 deals with systems for which the trajectories of the A-surfaces have constant curvature; all of these systems are found to be pseudospherical.

In another place‡ we have considered at length the A-surfaces with spherical lines of curvature in one system. Now we study the A-systems in which all the A-surfaces are of this kind; the general discussion of this problem and the solution of several particular cases form the substance of §10. In §11 those A-systems are discussed in which the surfaces in one family are spheres.

We have shown before§ that certain surfaces discovered by Bianchi are A-surfaces. In §12 we find the conditions to be satisfied in order that all the surfaces in one family of a triple system are surfaces of one of the three types given by Bianchi, and we determine the forms of the linear element of space when such a system is parametric.

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*l. c., p. 148. †l. c., p. 541.

‡Certain Surfaces with Plane or Spherical Lines of Curvature, American Journal, vol. 28, pp. 47–70.

§Surfaces analogous to the surfaces of Bianchi, Annali, Ser. III, vol. 12, pp. 113–143.

§1. *General Triply Orthogonal Systems of Surfaces.*

Consider space referred to a triply-orthogonal system of surfaces and write the linear element in the form

$$ds^2 = H_1^2 du_1^2 + H_2^2 du_2^2 + H_3^2 du_3^2. \tag{1}$$

Lame* has shown that the functions H_1, H_2, H_3 must satisfy six equations of condition, namely

$$\left\{\begin{aligned} \frac{\partial^2 H_1}{\partial u_2 \partial u_3} &= \frac{1}{H_2}\frac{\partial H_2}{\partial u_3}\frac{\partial H_1}{\partial u_2} + \frac{1}{H_3}\frac{\partial H_3}{\partial u_2}\frac{\partial H_1}{\partial u_3}, \\ \frac{\partial^2 H_2}{\partial u_3 \partial u_1} &= \frac{1}{H_3}\frac{\partial H_3}{\partial u_1}\frac{\partial H_2}{\partial u_3} + \frac{1}{H_1}\frac{\partial H_1}{\partial u_3}\frac{\partial H_2}{\partial u_1}, \\ \frac{\partial^2 H_3}{\partial u_1 \partial u_2} &= \frac{1}{H_1}\frac{\partial H_1}{\partial u_2}\frac{\partial H_3}{\partial u_1} + \frac{1}{H_2}\frac{\partial H_2}{\partial u_1}\frac{\partial H_3}{\partial u_2}, \end{aligned}\right. \tag{2}$$

and

$$\left\{\begin{aligned} &\frac{\partial}{\partial u_1}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial u_2}\right) + \frac{1}{H_3^2}\frac{\partial H_1}{\partial u_3}\frac{\partial H_2}{\partial u_3} = 0, \\ &\frac{\partial}{\partial u_2}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial u_3}\right) + \frac{1}{H_1^2}\frac{\partial H_2}{\partial u_1}\frac{\partial H_3}{\partial u_1} = 0, \\ &\frac{\partial}{\partial u_3}\left(\frac{1}{H_3}\frac{\partial H_1}{\partial u_3}\right) + \frac{\partial}{\partial u_1}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial u_1}\right) + \frac{1}{H_2^2}\frac{\partial H_3}{\partial u_2}\frac{\partial H_1}{\partial u_2} = 0. \end{aligned}\right. \tag{3}$$

Moreover, it can be shown† that these necessary conditions are also sufficient so that when three functions are known which satisfy them there is a corresponding triply orthogonal system and it is unique.

In the future discussion it will be necessary to have the expressions for the principal radii of these surfaces, and also for the first and second curvatures of their curves of intersection. If we denote by ρ_{31} the principal radius of curvature of the surface $u_3 = \text{const.}$ along the curve $u_2 = \text{const.}$ and similarly for the others, we have‡

$$\begin{aligned} \frac{1}{\rho_{12}} &= \frac{1}{H_1 H_2}\frac{\partial H_2}{\partial u_1}, & \frac{1}{\rho_{23}} &= \frac{1}{H_2 H_3}\frac{\partial H_3}{\partial u_2}, & \frac{1}{\rho_{31}} &= \frac{1}{H_3 H_1}\frac{\partial H_1}{\partial u_3}, \\ \frac{1}{\rho_{13}} &= \frac{1}{H_1 H_3}\frac{\partial H_3}{\partial u_1}, & \frac{1}{\rho_{21}} &= \frac{1}{H_2 H_1}\frac{\partial H_1}{\partial u_2}, & \frac{1}{\rho_{32}} &= \frac{1}{H_3 H_2}\frac{\partial H_2}{\partial u_3}, \end{aligned} \tag{4}$$

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* Leçons sur les coordonnes curvilignes, p. 76; Bianchi, Lezioni, II, p. 479.

† Ib.

‡ Bianchi, Lezioni, II, p. 482.

Again, if the first and second curvatures of the curves of parameter u_1 are denoted by $\frac{1}{R_1}$ and $\frac{1}{T_1}$, it is found that*

$$(5)\quad \begin{cases} \dfrac{1}{R_1^2} = \dfrac{1}{\rho_{21}^2} + \dfrac{1}{\rho_{31}^2}, \quad \dfrac{1}{R_2^2} = \dfrac{1}{\rho_{12}^2} + \dfrac{1}{\rho_{32}^2}, \quad \dfrac{1}{R_3^2} = \dfrac{1}{\rho_{23}^2} + \dfrac{1}{\rho_{13}^2}, \\ \dfrac{1}{T_1} = \dfrac{1}{H_1}\dfrac{\partial}{\partial u_1}\tan^{-1}\dfrac{\rho_{31}}{\rho_{21}}, \quad \dfrac{1}{T_2} = \dfrac{1}{H_2}\dfrac{\partial}{\partial u_2}\tan^{-1}\dfrac{\rho_{12}}{\rho_{32}}, \quad \dfrac{1}{T_3} = \dfrac{1}{H_3}\dfrac{\partial}{\partial u_3}\tan^{-1}\dfrac{\rho_{23}}{\rho_{13}}. \end{cases}$$

§2. *Systems with One Family of A-Surfaces.*

We have seen that for a surface S, whose linear element is written in the form

$$ds^2 = A^2du^2 + C^2dv^2$$

and the lines of curvature are parametric, to be an A-surface it is necessary and sufficient that the following equations be satisfied

$$(6)\quad \frac{1}{A}\frac{\partial C}{\partial u} = \frac{\partial \omega}{\partial u}, \quad \frac{1}{C}\frac{\partial A}{\partial v} = -\frac{\partial \omega}{\partial v}, \quad \frac{\partial^2 \omega}{\partial u^2} - \frac{\partial^2 \omega}{\partial v^2} = \sin\omega\cos\omega.$$

In this case the linear element of the spherical representation has the form

$$d\sigma^2 = \sin^2\omega\, du^2 + \cos^2\omega\, dv^2.$$

For such a surface to belong to the surfaces $u_3 = \text{const.}$ of a triple system ω must involve u_3 and from (6) it follows that A and C are determined only to within the same factor U_3, a function of u_3 alone. For each surface of the family this function would be a constant; and by replacing du and dv by $\frac{du}{U_3}$ and $\frac{dv}{U_3}$ respectively, this function would be made to disappear. Hence, the necessary and sufficient condition that the surfaces $u_3 = \text{const.}$ be A-surfaces is that H_1 and H_2 satisfy the conditions

$$(7)\quad -\frac{1}{H_2}\frac{\partial H_1}{\partial u_2} = \frac{\partial \omega}{\partial u_2}, \quad \frac{1}{H_1}\frac{\partial H_2}{\partial u_1} = \frac{\partial \omega}{\partial u_1},$$

where ω is a function of u_1, u_2, u_3, satisfying the equation

$$(8)\quad \frac{\partial^2 \omega}{\partial u_1^2} - \frac{\partial^2 \omega}{\partial u_2^2} = \frac{\sin\omega\cos\omega}{U_3^2},$$

U_3 being an arbitrary function of u_3, and those equations of condition (2) and (3) which are independent of the above.

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* Bianchi, l. c., p. 484.

The spherical representation of the lines of curvature of the surfaces $u_3 = \text{const.}$ is

$$(9) \qquad d\sigma^2 = \frac{\sin^2 \omega}{U_3^2}\, du_1^2 + \frac{\cos^2 \omega}{U_3^2}\, du_2^2.$$

From this and (1) we have the following expressions for the principal radii of the surfaces $u_3 = \text{const.}$

$$(10) \qquad \frac{1}{\rho_{31}} = -\frac{\sin \omega}{U_3 H_1}, \quad \frac{1}{\rho_{32}} = \frac{\cos \omega}{U_3 H_2}.$$

When these expressions are compared with (4), we find that we must have

$$(11) \qquad H_3 = -\frac{U_3}{\sin \omega} \frac{\partial H_1}{\partial u_3} = \frac{U_3}{\cos \omega} \frac{\partial H_2}{\partial u_3},$$

so that the following condition is to be satisfied

$$(12) \qquad \cos \omega \frac{\partial H_1}{\partial u_3} + \sin \omega \frac{\partial H_2}{\partial u_3} = 0.$$

In accordance with this we introduce a function ϕ defined by

$$(13) \qquad \frac{\partial H_1}{\partial u_3} = -\phi \sin \omega \frac{\partial \omega}{\partial u_3}, \quad \frac{\partial H_2}{\partial u_3} = \phi \cos \omega \frac{\partial \omega}{\partial u_3},$$

so that by quadratures and from (11) we have for the fundamental coefficients the following expressions

$$(14) \qquad \begin{cases} H_1 = \phi \cos \omega - \displaystyle\int \cos \omega \frac{\partial \phi}{\partial u_3}\, du_3 + F_1(u_1, u_2), \\ H_2 = \phi \sin \omega - \displaystyle\int \sin \omega \frac{\partial \phi}{\partial u_3}\, du_3 + F_2(u_1, u_2), \\ H_3 = \phi U_3 \dfrac{\partial \omega}{\partial u_3}, \end{cases}$$

where the functions F_1 and F_2 do not involve u_3.

The first of equations (2) can be put in either of the forms

$$\frac{\partial}{\partial u_3}\left(\frac{1}{H_2} \frac{\partial H_1}{\partial u_2}\right) = \frac{1}{H_2 H_3} \frac{\partial H_3}{\partial u_2} \frac{\partial H_1}{\partial u_3},$$

$$\frac{\partial}{\partial u_2}\left(\frac{1}{H_3} \frac{\partial H_1}{\partial u_3}\right) = \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial u_3} \frac{\partial H_1}{\partial u_2}.$$

If the preceding expressions be substituted in these equations, the former reduces to

$$(15)\qquad \frac{1}{H_2}\frac{\partial H_3}{\partial u_2}=\frac{U_3}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3},$$

and the latter vanishes identically. Again, the second of equations (2) takes either of the forms

$$\frac{\partial}{\partial u_3}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial u_1}\right)=\frac{1}{H_1H_3}\frac{\partial H_3}{\partial u_1}\frac{\partial H_2}{\partial u_3},$$
$$\frac{\partial}{\partial u_1}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial u_3}\right)=\frac{1}{H_1H_3}\frac{\partial H_1}{\partial u_3}\frac{\partial H_2}{\partial u_1},$$

of which the latter is satisfied identically in consequence of (11) and the former is reducible to

$$(16)\qquad \frac{1}{H_1}\frac{\partial H_3}{\partial u_1}=\frac{U_3}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}.$$

The third of equations (2) can be written in either of the forms

$$(17)\qquad \begin{aligned}\frac{\partial}{\partial u_2}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial u_1}\right)&=\frac{1}{H_1H_2}\frac{\partial H_2}{\partial u_1}\frac{\partial H_3}{\partial u_2},\\ \frac{\partial}{\partial u_1}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial u_2}\right)&=\frac{1}{H_1H_2}\frac{\partial H_1}{\partial u_2}\frac{\partial H_3}{\partial u_1},\end{aligned}$$

which by means of (15) and (16) can be given the forms

$$(18)\qquad \left\{\begin{aligned}\frac{\partial}{\partial u_2}\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)&=\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_2\partial u_3},\\ \frac{\partial}{\partial u_1}\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)&=-\frac{1}{\cos\omega}\frac{\partial\omega}{\partial u_2}\frac{\partial^2\omega}{\partial u_1\partial u_3}.\end{aligned}\right.$$

The first of equations (3) reduces to (8), and the second and third become in consequence of (7), (11), (15), and (16)

$$(19)\qquad \left\{\begin{aligned}\frac{\partial}{\partial u_1}\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)&=\frac{1}{U_3}\frac{\partial}{\partial u_3}\left(\frac{\sin\omega}{U_3}\right)+\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_2}\frac{\partial^2\omega}{\partial u_2\partial u_3},\\ \frac{\partial}{\partial u_2}\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)&=-\frac{1}{U_3}\frac{\partial}{\partial u_3}\left(\frac{\cos\omega}{U_3}\right)-\frac{1}{\cos\alpha}\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_1\partial u_3},\end{aligned}\right.$$

either of which is a consequence of the other and (8). We see then that ω must satisfy three equations, namely (8) and one each of (18) and (19).

From the preceding discussion it is seen that when ω satisfies these equations the only other conditions to be satisfied are equations (7) which become in consequence of (14)

$$(20)\qquad \begin{aligned} &\frac{\partial\phi}{\partial u_2}\cos\omega+\int\sin\omega\frac{\partial\omega}{\partial u_2}\frac{\partial\phi}{\partial u_3}du_3-\int\cos\omega\frac{\partial^2\phi}{\partial u_2\partial u_3}du_3+\frac{\partial F_1}{\partial u_2}\\ &\qquad\qquad+\frac{\partial\omega}{\partial u_2}\left[F_2-\int\sin\omega\frac{\partial\phi}{\partial u_3}du_3\right]=0,\\ &\frac{\partial\phi}{\partial u_1}\sin\omega-\int\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\phi}{\partial u_3}du_3-\int\sin\omega\frac{\partial^2\phi}{\partial u_1\partial u_3}du_3+\frac{\partial F_2}{\partial u_1}\\ &\qquad\qquad-\frac{\partial\omega}{\partial u_1}\left[F_1-\int\cos\omega\frac{\partial\phi}{\partial u_3}du_3\right]=0.\end{aligned}$$

If these equations are differentiated with respect to u_3, they become

$$(21)\qquad \begin{aligned} \sin\omega\frac{\partial\omega}{\partial u_3}\frac{\partial\phi}{\partial u_2}&=\left[F_2-\int\sin\omega\frac{\partial\phi}{\partial u_3}du_3\right]\frac{\partial^2\omega}{\partial u_2\partial u_3},\\ \cos\omega\frac{\partial\omega}{\partial u_3}\frac{\partial\phi}{\partial u_1}&=\left[F_1-\int\cos\omega\frac{\partial\phi}{\partial u_3}du_3\right]\frac{\partial^2\omega}{\partial u_1\partial u_3},\end{aligned}$$

which are merely equations (15) and (16) in another form.

Consider the particular case where ϕ is a constant which may be taken equal to unity. Equations (21) are satisfied if F_1 and F_2 are zero, and equations (20) vanish identically. The linear element (1) reduces then to

$$(22)\qquad ds^2=\cos^2\omega\,du_1^2+\sin^2\omega\,du_2^2+U_3^2\left(\frac{\partial\omega}{\partial u_3}\right)^2du_3^2,$$

from which it is seen that the surfaces $u_3=\text{constant}$ are pseudospherical. Bianchi has shown* that, when ω satisfies equations (8), (18), and (19), the above linear element defines the most general triply orthogonal systems of which the surfaces $u_3=\text{const.}$ are pseudospherical. Hence the theorem:

When one family of a triple system is composed of A-surfaces, the pseudospherical surfaces with the same spherical representation as the latter form a family of another triply orthogonal system.

Also,

Systems of lines on the unit sphere which are the images of the lines of curvature of pseudospherical surfaces forming a system of Lame, and these alone, are the spherical representation of the lines of curvature of A-surfaces forming such a system.

Such a triply-orthogonal system will be called an *A-system.*

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*Lezioni, II, p. 531.

When then there is such a family of lines, there is a great number of corresponding A-systems and their determination requires the solution of the equations (20).

As an example of the foregoing we inquire whether there is an A-system formed of parallels of pseudospherical surfaces. We have shown* that the parallels of such a surface are given by

$$H_1 = \cos\omega - a\sin\omega, \qquad H_2 = \sin\omega + a\cos\omega,$$

where a is a constant determining the various surfaces. For the case under consideration a may involve u_3. If the above expressions be substituted in equation (12), it reduces to

$$a\frac{\partial\omega}{\partial u_3} = 0.$$

From the expression (14) for H_3 it is seen that ω must involve u_3, so that the above equation is satisfied only when a is zero, that is, when the surfaces $u_3 =$ const. are pseudospherical.

§3. *Combescure-Darboux Transformations. Particular Solutions.*

From the foregoing discussion it follows that the determination of all the A-systems reduces to the finding of all pseudospherical systems and all systems related to the latter in such a way that the normals to the surfaces of the system are parallel to the normals to the corresponding surfaces of the pseudospherical system. Combescure and Darboux† have considered the problem of finding surfaces of Lame which have this correspondence with a given system, and we shall apply their results to our particular problem. We shall refer to it as the *Combescure-Darboux Transformation.*

With Darboux we put for the sake of brevity

$$\beta_{ij} = \frac{1}{H_i}\frac{\partial H_j}{\partial u_i}, \tag{23}$$

where i and j are different and assume the values 1, 2, 3. If we denote by X_1, Y_1, Z_1; $X_2 \ldots$; X_3, Y_3, Z_3, the direction-cosines of the tangents to the curves of parameter u_1, u_2, u_3 respectively, it can be shown that‡

$$\begin{aligned} \frac{\partial X_k}{\partial u_i} &= \beta_{ki}X_i, \\ \frac{\partial X_k}{\partial u_k} &= -\beta_{ik}X_i - \beta_{lk}X_l, \end{aligned} \tag{23'}$$

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* Surfaces with the same, etc., l. c. p. 123.

† Sur les surfaces orthogonales, Annales de L'Ecole Normale, ser. 1, Vol. 3, p. 114.

‡ Lame, l. c., p. 74; Bianchi, Lezioni, II, p. 494.

where i, k, l are different and take the values 1, 2, 3. In terms of the functions β the equations of Lame (2) and (3) are expressed as follows:

$$\frac{\partial \beta_{ki}}{\partial u_l} = \beta_{kl}\beta_{li},$$

$$\frac{\partial \beta_{ik}}{\partial u_i} + \frac{\partial \beta_{ki}}{\partial u_k} + \beta_{li}\beta_{lk} = 0.$$

If a triple system is given, the corresponding functions β are given by (23) and the determination of the transformation requires the solution for H_1, H_2, H_3 of the six equations (23) after the functions β have been given the calculated values. When the system is pseudospherical, the forms of H_1, H_2, H_3 are as given by (22), so that the determination of the Combescure-Darboux transforms of a pseudospherical system requires the solution of the equations

$$(24)\qquad \begin{array}{lll} \dfrac{1}{H_1}\dfrac{\partial H_2}{\partial u_1} = \dfrac{\partial \omega}{\partial u_1}, & \dfrac{1}{H_2}\dfrac{\partial H_3}{\partial u_2} = \dfrac{U_3}{\sin\omega}\dfrac{\partial^2 \omega}{\partial u_2 \partial u_3}, & \dfrac{1}{H_3}\dfrac{\partial H_1}{\partial u_3} = -\dfrac{\sin\omega}{U_3}, \\ \dfrac{1}{H_2}\dfrac{\partial H_1}{\partial u_2} = -\dfrac{\partial \omega}{\partial u_2}, & \dfrac{1}{H_3}\dfrac{\partial H_2}{\partial u_3} = \dfrac{\cos\omega}{U_3}, & \dfrac{1}{H_1}\dfrac{\partial H_3}{\partial u_1} = \dfrac{U_3}{\cos\omega}\dfrac{\partial^2 \omega}{\partial u_1 \partial u_3}. \end{array}$$

These are the very equations of condition which we have found by the first method.

If we eliminate H_1 and H_2 from the above equations, we are brought to the following triple system to be satisfied by H_3:

$$(25)\qquad \left\{ \begin{aligned} \frac{\partial^2 H_3}{\partial u_1 \partial u_2} &= \frac{V}{W}\frac{\partial \omega}{\partial u_1}\frac{\partial H_3}{\partial u_1} - \frac{W}{V}\frac{\partial \omega}{\partial u_2}\frac{\partial H_3}{\partial u_2}, \\ \frac{\partial^2 H_3}{\partial u_1 \partial u_3} &= \frac{\partial}{\partial u_3}\log\left(U_3 W\right)\frac{\partial H_3}{\partial u_1} - \sin\omega\, W H_3, \\ \frac{\partial^2 H_3}{\partial u_2 \partial u_3} &= \frac{\partial}{\partial u_3}\log\left(U_3 V\right)\frac{\partial H_3}{\partial u_2} + \cos\omega\, V H_3, \end{aligned} \right.$$

where for the sake of brevity we have put

$$(26)\qquad V = \frac{1}{\sin\omega}\frac{\partial^2 \omega}{\partial u_2 \partial u_3}, \qquad W = \frac{1}{\cos\omega}\frac{\partial^2 \omega}{\partial u_1 \partial u_3}.$$

In consequence of the equations (8), (18), and (19) the conditions of integrability of the above equations are satisfied, and it is readily seen that the general solution of this system involves three arbitrary functions. The only question then as to the existence of A-systems comes back to the question as to the existence of a function ω satisfying equations (8), (18), and (19). This problem has been attacked by Bianchi in seeking to establish the existence of pseudospherical systems of Lame and he has not only proved that they exist but has given an idea of the degree of arbitrariness in the expression for ω.*

When a solution H_3 of equations (25) is known, the corresponding functions H_1 and H_2 are given directly by (24). When the values of X_i, Y_i, Z_i are known, the cartesian coordinates of space referred to the system corresponding to a given set of solutions of (24) are found by quadratures; namely

$$dx = \Sigma H_i X_i \, du_i, \qquad dy = \Sigma H_i Y_i \, du_i, \qquad dz = \Sigma H_i Z_i \, du_i.$$

If we substitute the expression for H_3 given in (14) in equations (25), they can be reduced to

$$(27)\quad \left\{ \begin{aligned} & V \cdot W \cdot \frac{\partial \omega}{\partial u_3} \frac{\partial^2 \phi}{\partial u_1 \partial u_2} + V^2 \left(\sin \omega \; W - \frac{\partial \omega}{\partial u_1} \frac{\partial \omega}{\partial u_3} \right) \frac{\partial \phi}{\partial u_1} \\ & \qquad\qquad + W^2 \left(\cos \omega \; V + \frac{\partial \omega}{\partial u_2} \frac{\partial \omega}{\partial u_3} \right) \frac{\partial \phi}{\partial u_2} = 0, \\ & \frac{\partial^2 \phi}{\partial u_1 \partial u_3} + \frac{\partial}{\partial u_3} \log \frac{\frac{\partial \omega}{\partial u_3}}{W} \frac{\partial \phi}{\partial u_1} + \frac{\cos \omega \; W}{\frac{\partial \omega}{\partial u_3}} \frac{\partial \phi}{\partial u_3} = 0, \\ & \frac{\partial^2 \phi}{\partial u_2 \partial u_3} + \frac{\partial}{\partial u_3} \log \frac{\frac{\partial \omega}{\partial u_3}}{V} \frac{\partial \phi}{\partial u_2} + \frac{\sin \omega \; V}{\frac{\partial \omega}{\partial u_3}} \frac{\partial \phi}{\partial u_3} = 0. \end{aligned} \right.$$

These equations can be derived readily from equations (20) and (21). We shall consider now several particular cases for which equations (27) take more simple forms.

It is seen that the above equations are satisfied when ϕ is a constant, and it was seen that if F_1 and F_2 are zero at the same time the system is pseudospherical. We inquire whether the latter is the only solution of this kind. For ϕ, a constant, equations (21) reduce to

$$F_1 \frac{\partial^2 \omega}{\partial u_1 \, \partial u_3} = 0, \qquad F_2 \frac{\partial^2 \omega}{\partial u_2 \, \partial u_3} = 0.$$

From (18) it follows that if $\frac{\partial^2 \omega}{\partial u_2 \partial u_3}$ is zero so also is $\frac{\partial^2 \omega}{\partial u_1 \partial u_3}$ or $\frac{\partial \omega}{\partial u_2}$. The former leads by means of (19) to

$$\frac{\partial}{\partial u_3} \frac{\cos \omega}{U_3} = \frac{\partial}{\partial u_3} \frac{\sin \omega}{U_3} = 0,$$

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* Lezioni, II, p. 532 et seq.

from which we have

$$\cos\omega = U_3 f_1(u_1, u_2), \qquad \sin\omega = U_3 f_2(u_2, u_3),$$

where the functions f_1 and f_2 do not involve u_3. This shows that U_3 is a constant and then ω would be independent of u_3, which we have seen to be impossible. The only other possibility then is that F_1 and $\frac{\partial\omega}{\partial u_2}$ are zero, in which case F_3 is an arbitrary function of u_2 alone, as is seen from (20). Hence the theorem:

The function ϕ is constant for pseudospherical systems and for certain systems for which F_1 or F_2 is zero, and F_3 a function of u_2 alone, or F_1 a function of u_1 alone respectively; and these are the only cases.

We inquire now whether ϕ can be a function of u_3 alone. From the second and third of equations (27) we see that in this case we must have

$$\frac{\partial^2\omega}{\partial u_1\,\partial u_3} = \frac{\partial^2\omega}{\partial u_2\,\partial u_3} = 0.$$

This has been seen to be impossible, hence *the function ϕ must involve one of the parameters u_1, u_2* or *be a constant.*

This suggests finally the inquiry as to whether ϕ can be independent of u_2. From the third of equations (27) we have

$$\frac{\partial^2\omega}{\partial u_2\,\partial u_3} = 0,$$

in consequence of which the first of these equations is satisfied identically and ϕ must be an integral of the second. We have seen that the above condition is equivalent to the vanishing of $\frac{\partial\omega}{\partial u_2}$. The converse also is true, so that we have the theorem:

If one of the functions ϕ and ω is independent of u_1 or u_2, so also is the other.

§4. *Generalized Bäcklund Transformations of A-Systems.*

In our previous discussion* of A-surfaces we have established a transformation of these surfaces which reduces for the pseudospherical surfaces to the Bäcklund transformation. It may be defined as follows: Given an A-surface and a solution θ of the system

$$\sin\sigma\left(\frac{\partial\theta}{\partial u_1} + \frac{\partial\omega}{\partial u_2}\right) = \sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega,$$

$$\sin\sigma\left(\frac{\partial\theta}{\partial u_2} + \frac{\partial\omega}{\partial u_1}\right) = -\cos\theta\sin\omega + \cos\sigma\sin\theta\cos\omega,$$

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* Surfaces with the same etc., l. c. p. 148.

where σ is a constant. In the tangent plane to S draw the line through the point of contact which makes the angle θ with the tangent to the curve $u_2 =$ const. through the point. The plane which cuts the tangent plane along this line and under the angle σ envelopes another A-surface for which the spherical representation of the lines of curvature is

$$d\sigma_1^2 = \sin^2\theta \, du_1^2 + \cos^2\theta \, du_2^2.$$

We are now going to consider an A-system and determine the conditions to be satisfied in order that the transforms of the A-surfaces form part of a second triple system.

Just as equations (6) had to be given a more general form, namely (7) and (8), in the case of triple systems, so likewise the equations for the determination of θ take the form

$$\begin{aligned} \varkappa\left(\frac{\partial\theta}{\partial u_1} + \frac{\partial\omega}{\partial u_2}\right) &= \sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega, \\ \varkappa\left(\frac{\partial\theta}{\partial u_2} + \frac{\partial\omega}{\partial u_1}\right) &= -\cos\theta\sin\omega + \cos\sigma\sin\theta\cos\omega, \end{aligned} \tag{28}$$

where we have put

$$\varkappa = U_3 \sin\sigma,$$

from which it follows that at most $\varkappa$ is a function of u_3. We denote by x_1, y_1, z_1 the coordinates of the point of tangency on the transform, by μ the distance from this point to the line of intersection of the two planes and by λ the distance from the foot of this perpendicular to the point of contact of the given surface S. From the expressions for λ and μ as found in the case of the transformation of a single surface* we have for the present case

$$\begin{aligned} \lambda &= \varkappa\,(H_1\cos\omega + H_2\sin\omega), \\ \mu &= \varkappa\,(-H_1\sin\omega + H_2\cos\omega). \end{aligned}$$

Hence we have for the coordinates of the transform

$$x_1 = x + (\lambda\cos\theta - \mu\cos\sigma\sin\theta)\,X_1 + (\lambda\sin\theta + \mu\cos\sigma\cos\theta)\,X_2 - \mu\sin\sigma X_3$$

and analogous expressions for y_1 and z_1, X_1, X_2, X_3 having the interpretation previously defined. From (23′) and the expressions for the functions β given by (23) and (24) we find readily

$$\begin{aligned} &\frac{\partial X_1}{\partial u_1} = \frac{\partial\omega}{\partial u_2}X_2 + \frac{\sin\omega}{U_3}X_3, && \frac{\partial X_1}{\partial u_2} = \frac{\partial\omega}{\partial u_1}X_2, && \frac{\partial X_1}{\partial u_3} = \frac{1}{H_1}\frac{\partial H_3}{\partial u_1}X_3; \\ &\frac{\partial X_2}{\partial u_1} = -\frac{\partial\omega}{\partial u_2}X_1, && \frac{\partial X_2}{\partial u_2} = -\frac{\partial\omega}{\partial u_1}X_1 - \frac{\cos\omega}{U_3}X_3, && \frac{\partial X_2}{\partial u_3} = \frac{1}{H_2}\frac{\partial H_3}{\partial u_2}X_3; \\ &\frac{\partial X_3}{\partial u_1} = -\frac{\sin\omega}{U_3}X_1, && \frac{\partial X_3}{\partial u_2} = \frac{\cos\omega}{U_3}X_2, && \frac{\partial X_3}{\partial u_3} = -\frac{1}{H_1}\frac{\partial H_3}{\partial u_1}X_1 - \frac{1}{H_2}\frac{\partial H_3}{\partial u_2}X_2. \end{aligned}$$

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[*] l. c. p. 150.

If we make use of these relations in our calculation of the differential quotients of x_1 and for the sake of brevity put

$$p=\frac{\partial H_1}{\partial u_1}+H_2\frac{\partial\omega}{\partial u_1},\qquad q=\frac{\partial H_2}{du_2}-H_1\frac{\partial\omega}{\partial u_2},$$

we find

$$(29)\left\{\begin{aligned}
\frac{\partial x_1}{\partial u_1}&=\frac{1}{\kappa}\left[\kappa^2 p+\lambda\cos\theta-\mu\cos\sigma\sin\theta\right]\Big[(\cos\theta\cos\omega+\cos\sigma\sin\theta\sin\omega)X_1\\
&\qquad+(\sin\theta\cos\omega-\cos\sigma\cos\theta\sin\omega)X_2+\sin\sigma\sin\omega X_3\Big],\\
\frac{\partial x_1}{\partial u_2}&=\frac{1}{\kappa}\left[\kappa^2 q+\lambda\sin\theta+\mu\cos\sigma\cos\theta\right]\Big[(\cos\theta\sin\omega-\cos\sigma\sin\theta\cos\omega)X_1\\
&\qquad+(\sin\theta\sin\omega+\cos\sigma\cos\theta\cos\omega)X_2-\sin\sigma\cos\omega X_3\Big],\\
\frac{\partial x_1}{\partial u_3}&=\Big[(\lambda\sin\theta\cos\sigma+\mu\cos\theta)\frac{\partial\omega}{\partial u_3}-\kappa\cos\sigma\sin\theta\frac{H_3}{U_3}\\
&\quad-(\lambda\sin\theta+\mu\cos\sigma\cos\theta)\frac{\partial\theta}{\partial u_3}+(\lambda\cos\theta-\mu\cos\sigma\sin\theta)\frac{\partial\log\kappa}{\partial u_3}\\
&\quad+\frac{\mu\sin\sigma}{H_1}\frac{\partial H_3}{\partial u_1}\Big]X_1+\Big[-(\lambda\cos\theta\cos\sigma-\mu\sin\theta)\frac{\partial\omega}{\partial u_3}+\kappa\cos\sigma\cos\theta\frac{H_3}{U_3}\\
&\quad+(\lambda\cos\theta-\mu\cos\sigma\sin\theta)\frac{\partial\theta}{\partial u_3}+(\lambda\sin\theta+\mu\cos\sigma\cos\theta)\frac{\partial\log\kappa}{\partial u_3}\\
&\quad+\frac{\mu\sin\sigma}{H_2}\frac{\partial H_3}{\partial u_2}\Big]X_2+\Big[H_3\cos^2\sigma+(\lambda\cos\theta-\mu\cos\sigma\sin\theta)\frac{1}{H_1}\frac{\partial H_2}{\partial u_1}\\
&\quad+(\lambda\sin\theta+\mu\cos\sigma\cos\theta)\frac{1}{H_2}\frac{\partial H_1}{\partial u_2}+\lambda\sin\sigma\frac{\partial\omega}{\partial u_3}-\mu\sin\sigma\frac{\partial\log\kappa}{\partial u_3}\Big]X_3.
\end{aligned}\right.$$

From these expressions it is readily found that

$$\Sigma\frac{\partial x_1}{\partial u_1}\frac{\partial x_1}{\partial u_2}=0,$$

and by direct calculation we find that

$$\begin{aligned}\sum\frac{\partial x_1}{\partial u_1}\frac{\partial x_1}{\partial u_3}=H_2(p\kappa^2+\lambda\cos\theta-\mu\cos\sigma\cos\theta)\Big[\frac{\partial\omega}{\partial u_3}-\cos\sigma\frac{\partial\theta}{\partial u_3}+\frac{\sin\sigma\cos\theta}{H_1}\frac{\partial H_3}{\partial u_1}\\
+\frac{\sin\sigma\sin\theta}{H_2}\frac{\partial H_3}{\partial u_2}-\frac{H_2}{H_1}\frac{\partial\log\kappa}{\partial u_3}\Big],\end{aligned}$$

$$\begin{aligned}\sum\frac{\partial x_1}{\partial u_2}\frac{\partial x_1}{\partial u_3}=-H_1(q\kappa^2+\lambda\sin\theta+\mu\cos\sigma\sin\theta)\Big[\frac{\partial\omega}{\partial u_3}-\cos\sigma\frac{\partial\theta}{\partial u_3}+\frac{\sin\sigma\cos\theta}{H_1}\frac{\partial H_3}{\partial u_1}\\
+\frac{\sin\sigma\sin\theta}{H_2}\frac{\partial H_3}{\partial u_2}+\frac{H_1}{H_2}\frac{\partial\log\kappa}{\partial u_3}\Big].\end{aligned}$$

From these expressions and the above for $\frac{\partial x_1}{\partial u_1}$ and $\frac{\partial x_1}{\partial u_2}$ it follows that for the transformed system to be triply orthogonal it is necessary that the expressions in the brackets vanish. These equations lead to

$$\frac{\partial \log \kappa}{\partial u_3} = 0,$$

from which it follows that κ is a constant, and to

$$\cos\sigma \frac{\partial \theta}{\partial u_3} = \frac{\sin\sigma \cos\theta}{H_1}\frac{\partial H_3}{\partial u_1} + \frac{\sin\sigma \sin\theta}{H_2}\frac{\partial H_3}{\partial u_2} + \frac{\partial \omega}{\partial u_3}$$

which must be satisfied by θ in addition to (28). By means of equations (15) and (16) this equation can be given the form

$$\text{(30)} \qquad \cos\sigma \frac{\partial \theta}{\partial u_3} = \frac{\kappa \cos\theta}{\cos\omega}\frac{\partial^2 \omega}{\partial u_1 \partial u_3} + \frac{\kappa \sin\theta}{\sin\omega}\frac{\partial^2 \omega}{\partial u_2 \partial u_3} + \frac{\partial \omega}{\partial u_3}.$$

It is important to note that equations (28) and (30) involve only ω, σ, κ, and hence θ does not vary with the Combescure-Darboux transformation. We have then the theorem:

Whenever a function θ is known which transforms a given A-system into a similar system, the same function determines a like transformation for all the Combescure-Darboux transforms of the given system.

In consequence of (30) the expression for $\frac{\partial x_1}{\partial u_3}$ can be reduced to

$$\frac{\partial x_1}{\partial u_3} = \left[\lambda \sin\sigma \frac{\partial \theta}{\partial u_3} + \cos\sigma\, H_3 + (\lambda\cos\theta\cos\sigma - \mu\sin\theta)\frac{U_3}{\cos\omega}\frac{\partial^2\omega}{\partial u_1 \partial u_3} + (\lambda\sin\theta\cos\sigma \right.$$
$$\left. + \mu\cos\theta)\frac{U_3}{\sin\omega}\frac{\partial^2\omega}{\partial u_2 \partial u_3}\right](-\sin\sigma\sin\theta\, X_1 + \sin\sigma\cos\theta\, X_2 + \cos\sigma\, X_3).$$

For the linear element of space referred to the transformed system we find from the above

$$ds_1^2 = \frac{1}{\kappa^2}(\kappa^2 p + \lambda\cos\theta - \mu\cos\sigma\sin\theta)^2 du_1^2 + \frac{1}{\kappa^2}(\kappa^2 q + \lambda\sin\theta + \mu\cos\sigma\cos\theta)^2 du_2^2$$
$$+ \left[\lambda\sin\sigma\frac{\partial\theta}{\partial u_3} + \cos\sigma\, H_3 + (\lambda\cos\theta\cos\sigma - \mu\sin\theta)\frac{U_3}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right.$$
$$\left. + (\lambda\sin\theta\cos\sigma + \mu\cos\theta)\frac{U_3}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right]^2 du_3^2,$$

which gives for the surfaces $u_3 = \text{const.}$ the expression for the linear element which we have found before.

The system (28) and (30), which must be satisfied by θ, can be replaced by the total differential equation

$$\begin{aligned} d\theta = & \left(\frac{\sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega}{\varkappa} - \frac{\partial\omega}{\partial u_2}\right) du_1 \\ & - \left(\frac{\cos\theta\sin\omega - \cos\sigma\sin\theta\cos\omega}{\varkappa} + \frac{\partial\omega}{\partial u_1}\right) du_2 \\ & + \frac{\varkappa}{\cos\sigma}\left(\frac{\cos\theta}{\cos\omega}\frac{\partial^2\omega}{\partial u_1 \partial u_3} + \frac{\sin\theta}{\sin\omega}\frac{\partial^2\omega}{\partial u_2 \partial u_3} + \frac{1}{\varkappa}\frac{\partial\omega}{\partial u_3}\right). \end{aligned} \tag{31}$$

Since ω satisfies equations (8), (18), and (19), the conditions of integrability of this equation are satisfied, so that for each value of $\varkappa$ there is a general integral involving an arbitrary constant; hence corresponding to each value of $\varkappa$ there is an infinity of transformations of a given A-system into similar systems.

Bianchi[*] has called attention to the fact that if we introduce a new dependent variable in (31), defined by

$$\tan\frac{\theta}{2} = \Lambda,$$

the equation in Λ is of the Riccati form and consequently when a particular integral is known the equation can be integrated by quadratures. From this it follows that if a transform of a given system is known, all the transforms can be found by quadratures. But the latter admit the given system for a transform, and hence when the transforms of a given systen are known the transforms of the former are given by quadratures. We shall find furthermore that even these quadratures are unnecessary.

We have shown that if a surface of Bonnet S is transformed by means of θ_1 and σ_1 into S_1 and by means of θ_2 and σ_2 into S_2, there is a function ϕ given by

$$\tan\frac{\phi - \omega}{2} = \frac{\sin\dfrac{\sigma_1 + \sigma_2}{2}}{\sin\dfrac{\sigma_1 - \sigma_2}{2}} \tan\frac{\theta_1 - \theta_2}{2},$$

by means of which and σ_1 the surface S_1 can be transformed into a surface S_3, and S_2 can be transformed into the same surface S_3 by means of ϕ and σ_1. Consider now an A-system Σ and the associated pseudospherical system P. Let θ_1 be a particular solution of equation (31) for $\sigma = \sigma_1$ and θ_2 a particular solution for $\sigma = \sigma_2$. The former enables us to transform Σ into another A-system Σ_1 and

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[*] Lezioni, II, p. 540.

P into the associated pseudospherical system P_1; in like manner we pass from Σ by means of θ_2 to an A-system Σ_2 and from P to the associated pseudospherical system P_2. If the surfaces of Σ_1 are transformed by means of ϕ and σ_2 we get a family Σ_3 which is the same as is obtained from Σ_2 by means of ϕ and σ_1. In a similar manner we get from P_1 and P_2 the same pseudospherical system P_3. But Bianchi has shown* that the system P_3 is a family of Lame, and we have shown that the function which transforms one family of Lame of A-surfaces into a family of this type leads to a similar transformation of all the Combescure-Darboux transforms of the given family. Hence we have the following theorem of permutability for A-systems:

If Σ_1 *and* Σ_2 *are two triply orthogonal A-systems bound to a system* Σ *by two transformations of Bäcklund of constants* x_1 *and* x_2, *there is a transformed system of* B_1 *of constant* x_2 *which coincides with a transformed system of* B_2 *of constant* x_1.

From this result we pass at once to the following theorem:

If one knows how to find all the transforms of an A-system for all values of x, *the determination of the transforms of the latter requires only algebraic processes and differentiation.*

§5. *Weingarten Systems of A-Surfaces.*

When in particular the function U_3 is a constant, which without loss of generality can be taken as unity,† the equations (8), (18), and (19) reduce to

$$(32)\qquad\left\{\begin{aligned}
&\frac{\partial^2\omega}{\partial u_1^2}-\frac{\partial^2\omega}{\partial u_2^2}=\sin\omega\cos\omega,\\
&\frac{\partial}{\partial u_1}\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)=\cos\omega\frac{\partial\omega}{\partial u_3}+\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_2}\frac{\partial^2\omega}{\partial u_2\partial u_3},\\
&\frac{\partial}{\partial u_2}\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)=\sin\omega\frac{\partial\omega}{\partial u_3}-\frac{1}{\cos\omega}\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_1\partial u_3},\\
&\frac{\partial^3\omega}{\partial u_1\partial u_2\partial u_3}=\cot\omega\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_2\partial u_3}-\tan\omega\frac{\partial\omega}{\partial u_2}\frac{\partial^2\omega}{\partial u_1\partial u_3}.
\end{aligned}\right.$$

Bianchi‡ has shown that this system may be replaced by the equivalent system

$$(33)\qquad\left\{\begin{aligned}
&\frac{\partial^2\omega}{\partial u_1^2}-\frac{\partial^2\omega}{\partial u_2^2}=\sin\omega\cos\omega,\\
&\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)^2+\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)^2-\left(\frac{\partial\omega}{\partial u_3}\right)^2=F(u_3),
\end{aligned}\right.$$

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* Lezioni, II, p. 541.

† Bianchi shows, l. c., p. 528, that the total curvature of one of the surfaces $u_3=$ const. is $^{-1}/R^2$.

‡ Lezioni, II, p. 550.

where $F(u_3)$ is a function of u_3 alone. Three cases present themselves according as this function is positive, zero, or negative. As no restriction has been placed thus far upon the parameter u_3, it follows that it can be so chosen that we have the three cases

$$(34) \qquad F(u_3) = +1, \qquad F(u_3) = 0, \qquad F(u_3) = -1.$$

From (4) and (22) it follows that, for all of these cases, when the surfaces $u_3 = \text{const.}$ are pseudospherical we have

$$\frac{1}{\rho_{31}} = -\tan\omega, \qquad \frac{1}{\rho_{32}} = \cot\omega$$

and consequently all the surfaces $u_3 = \text{const.}$ have the same total curvature. Since Weingarten was the first to discover the existence of pseudospherical systems with this property, Bianchi has called them the *systems of Weingarten.* In consequence of this we shall refer to all systems for which U_3 is constant as *Weingarten A-systems.*

Bianchi[*] has shown that a Bäcklund transformation of a system of Weingarten of any one of the three types (34) leads to a system of Weingarten of the same type. Since all of this refers only to the spherical representation we have the theorem:

Bäcklund transformations of an A-system of any of the Weingarten types give systems of the same kind.

If we denote by θ the angle which the principal normal to a curve of parameter u_3, drawn in the positive direction, at the point where it meets S makes with the positive direction of the tangent to the curve $u_2 = \text{const.}$ on S through the point, it is found that

$$\cos\theta = -\frac{R_3}{H_1}\frac{\partial}{\partial u_1}\log H_3,$$

$$\sin\theta = -\frac{R_3}{H_2}\frac{\partial}{\partial u_2}\log H_3,$$

which, in consequence of (15) and (16), become for the case where $F(u_3)$ is zero in (33)

$$\cos\theta\frac{\partial\omega}{\partial u_3} = -\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}, \qquad \sin\theta\frac{\partial\omega}{\partial u_3} = -\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}.$$

Since ω must satisfy equations (18), we get that θ must satisfy the system

$$(34') \qquad \begin{aligned} \frac{\partial\omega}{\partial u_1} + \frac{\partial\theta}{\partial u_2} &= -\cos\theta\sin\omega, \\ \frac{\partial\omega}{\partial u_2} + \frac{\partial\theta}{\partial u_1} &= \sin\theta\cos\omega, \end{aligned}$$

[†] Ib., p. 562.

and from the preceding we have

$$\frac{\cos\theta}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}+\frac{\sin\theta}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}+\frac{\partial\omega}{\partial u_3}=0.$$

But these are the forms which equations (28) and (30) take for $\varkappa$ equal to unity and σ a right angle. Again, since $F(u_3)$ is zero, the second of equations (33) is of such a form that there is only one value of θ satisfying the above equations. Hence the theorem:

The osculating planes of the curves of parameter u_3, *at the intersections of the latter with the surfaces* $u_3 = const.$, *envelope complementary transforms of the surfaces* $u_3 = const.$ *and form the only system of Lamé composed of such transforms of the given system.*

Since it was seen that $\varkappa$ in (28) is a constant, it follows that for σ to be a right angle U_3 must be constant. We have just seen that when F is zero there is only one transformation of this kind. In a similar manner it can be shown that when F is positive there are two transformations, and none when F is negative.

§6. *Systems of Moulure Surfaces. Lines of Equidistance.*

We have seen that, when ω does not involve u_2, ϕ also is independent of this parameter. We shall now consider this case.

Equations (8), (18), and (19) which determine ω reduce to

$$\frac{\partial^2\omega}{\partial u_1^2}-\frac{\partial^2\omega}{\partial u_2^2}=\frac{\sin\omega\cos\omega}{U_3^2},$$

$$\frac{1}{U_3}\frac{\partial}{\partial u_3}\left(\frac{\cos\omega}{U_3}\right)+\frac{1}{\cos\omega}\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}=0,$$

$$\frac{\partial}{\partial u_1}\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right)=\frac{1}{U_3}\frac{\partial}{\partial u_3}\left(\frac{\sin\omega}{U_3}\right),$$

of which the last is a direct consequence of the first two and these can be replaced by

$$\left(\frac{\partial\omega}{\partial u_1}\right)^2+\frac{\cos^2\omega}{U_3^2}=c, \tag{35}$$

where c is a constant.

The general integral of this equation is

$$\cos\omega=\operatorname{sn}(\tau,\varkappa)$$

where

$$\tau=\frac{u_1}{a}+f(u_3),\qquad \varkappa=\frac{a}{U_3},$$

the functions f and U_3 being arbitrary functions of u_3 and a an arbitrary constant. From (7) and (15) it follows that H_1 and H_3 are independent of u_2, and hence from (14) it is seen that F_1 is a function of u_1 alone. The first and last of equations (27) for the determination of ϕ are satisfied identically and the second reduces to

$$\frac{\frac{\partial\omega}{\partial u_3}}{W}\frac{\partial^2\phi}{\partial u_1\,\partial u_3}+\frac{\partial}{\partial u_3}\left(\frac{\frac{\partial\omega}{\partial u_3}}{W}\right)\frac{\partial\phi}{\partial u_1}+\cos\omega\,\frac{\partial\phi}{\partial u_3}=0. \tag{36}$$

For each integral of this equation the corresponding functions F_1 and F_2 are given by (20) and (21) and then the triple system is completely determined.

Since ω does not involve u_2, the spherical images of the lines of curvature $u_2=$ const. on the surfaces $u_3=$ const. are great circles with a common diameter and consequently the latter are *moulure* surfaces, and for the surfaces the curves $u_2=$ const. are geodesics, which is evident analytically owing to the fact that H_1 does not involve u_2. Moreover, the functions H_3 are independent of u_2 and consequently the lines of curvature $u_2=$ const. on the surfaces $u_1=$ const. are geodesics. Again, from the general expressions (4) for the principal radii of the surfaces $u_2=$ const. it is seen that the latter surfaces are planes—the planes of the lines of curvature in one system on the surfaces $u_3=$ const. and $u_1=$ const. From this it follows that the latter surfaces also are moulure surfaces with the same cylindrical generator as the surfaces $u_3=$ const. Since the most general moulure surface is generated by a plane curve whose plane rolls without sliding over a cylinder, it is evident that the most general triple system with a family of moulure surfaces is obtained by tracing a family of curves and their orthogonal trajectories in a plane and rolling the plane over the cylinder, thus generating two families of moulure surfaces which together with the different positions of the plane form a triply orthogonal system.

Consider now the case of Weingarten systems of moulure surfaces; without loss of generality we can put U_3 and c equal to unity in (35), in which case the latter reduces to

$$\frac{\partial\omega}{\partial u_1}=\sin\omega, \tag{37}$$

which can readily be integrated giving

$$\tan\frac{\omega}{2}=e^{u_1+f(u_3)}, \tag{38}$$

where f is an arbitrary function of u_3; but as no particular choice has been made of the parameter u_3 in deriving the general equations (8), (18), and (19), it follows that in all generality this function of u_3 can be replaced by u_3 alone. When this value for ω is substituted in equation (36), the latter becomes

$$\frac{\partial}{\partial u_1}\left(\sin\omega\frac{\partial\phi}{\partial u_3}\right)=0,$$

of which the integral is

$$\phi=\int\frac{\chi(u_3)}{\sin\omega}\,du_3+\psi(u_1),$$

where χ and ψ are arbitrary functions. When this substitution is made in the second of equations (21), it is found that

$$F_1=\frac{\partial\psi}{\partial u_1}$$

and from the first of equations (20) it follows that F_2 is an arbitrary function of u_2. Gathering together these results we have the following expressions for the fundamental coefficients of a general Weingarten system of moulure surfaces:

$$(39)\qquad \begin{cases} H_1=\cos\omega\left[\int\frac{\chi\,du_3}{\sin\omega}+\psi\right]-\int\frac{\chi\cos\omega}{\sin\omega}du_3+\psi, \\ H_2=\sin\omega\left[\int\frac{\chi\,du_3}{\sin\omega}+\psi\right]-\int\chi\,du_3+F_2(u_2), \\ H_3=\left[\int\frac{\chi\,du_3}{\sin\omega}+\psi\right]\frac{\partial\omega}{\partial u_3}, \end{cases}$$

where ψ, F_2, and χ are arbitrary functions of u_1, u_2, and u_3 respectively.

We have shown elsewhere* that the linear element of a moulure surface can be given the form

$$ds^2=U_1^2du_1^2+\left[U_2-\int U_1\cos u_1\,du_1\right]du_2^2,$$

where U_1 and U_2 are functions of u_1 and u_2 respectively and the function *cos* u_1 arises from the choice of the spherical representation. At the same time it was found that the function U_2 determines the character of the cylindrical generator and U_1 the form of the generating curve of the surface. Comparing this result with (39), we note that the functions χ and ψ determine the families of curves generating the two families of moulure surfaces and F_2 fixes the form of the cylinder.

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*Surfaces whose lines of curvature in one system are represented on the sphere by great circles. Amer. Journ. Vol. 25, p. 359.

In consequence of equations (15) and (16), the second of equations (33) can be put in the form

$$\left(\frac{1}{H_1}\frac{\partial H_3}{\partial u_1}\right)^2+\left(\frac{1}{H_2}\frac{\partial H_3}{\partial u_2}\right)^2-\frac{H_3^2}{\phi_2}=F(u_3),$$

or

$$\Delta H_3=F(u_3)+\frac{H_3^2}{\phi^2}, \tag{40}$$

where ΔH_3 denotes the first Beltrami differential parameter formed with respect to the linear element of the surfaces $u_3=$ const.

$$ds^2=H_1^2du_1^2+H_2du_2^2.$$

Upon these surfaces there is a family of curves, defined by the equation $H_3=$ const., which have the property, as is seen from (1), that along such a curve the distances to the next surface of the system are constant; on this account they are called the curves of equidistance. From the theory of geodesic lines* and equation (40), it follows that the curves of equidistance on the surfaces $u_3=$ const. are geodesic parallels when ϕ is a constant or a function of H_3, or what is the same thing as the latter when ϕ is a function of $\frac{\partial\omega}{\partial u_3}$. Consider now the case

$$\phi=f\left(\frac{\partial\omega}{\partial u_3}\right),$$

whereupon equations (21) become

$$\begin{aligned}\sin\omega\frac{\partial\omega}{\partial u_3}\frac{\partial^2\omega}{\partial u_2\partial u_3}f'&=\left(F_2-\int\sin\omega\frac{\partial\phi}{\partial u_3}du_3\right)\frac{\partial^2\omega}{\partial u_2\partial u_3},\\ \cos\omega\frac{\partial\omega}{\partial u_3}\frac{\partial^2\omega}{\partial u_1\partial u_3}f'&=\left(F_1-\int\cos\omega\frac{\partial\phi}{\partial u_3}du_3\right)\frac{\partial^2\omega}{\partial u_1\partial u_3}.\end{aligned}\tag{41}$$

If neither $\frac{\partial^2\omega}{\partial u_1\partial u_3}$ nor $\frac{\partial^2\omega}{\partial u_2\partial u_3}$ is zero, these equations reduce to forms from which f' can be eliminated giving

$$\tan\omega\left(F_1-\int\cos\omega\frac{\partial\phi}{\partial u_3}du_3\right)=F_2-\int\sin\omega\frac{\partial\phi}{\partial u_3}du_3.$$

Differentiate with respect to u_3; this gives

$$\sec^2\omega\frac{\partial\omega}{\partial u_3}\left(F_1-\int\cos\omega\frac{\partial\phi}{\partial u_3}du_3\right)=0,$$

which is possible only in case ϕ is constant and F_1 is zero, which leads by the former equation to the necessity of the vanishing of F_2. We have seen that these conditions characterize the pseudospherical systems of Weingarten. We have seen that both $\frac{\partial^2\omega}{\partial u_1\partial u_3}$ and $\frac{\partial^2\omega}{\partial u_2\partial u_3}$ cannot be zero, so it only remains to consider the case where the latter vanishes; then $\frac{\partial\omega}{\partial u_2}$ vanishes, and consequently the surfaces $u_3=\text{const.}$ are moulure surfaces. The second of equations (41) becomes

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*Bianchi, Lezioni, I, p. 198.

$$\cos\omega\frac{\partial\omega}{\partial u_3}f'=F_1-\int\cos\omega\frac{\partial\phi}{\partial u_3}du_3.$$

Differentiate with respect to u_3; the result may be written

$$\frac{\partial^2\omega}{\partial u_3^2}\left(\frac{2f'+f''\frac{\partial\omega}{\partial u_3}}{f'\frac{\partial\omega}{\partial u_3}}\right)=\tan\omega\frac{\partial\omega}{\partial u_3},$$

where primes denote differentiation with respect to the argument. This equation can be integrated with respect to u_3 with the result

$$\Psi\left(\frac{\partial\omega}{\partial u_3}\right)=-\log\lambda\cos\omega,$$

where the form of the function Ψ is not essential and λ is a function independent of u_3; this equation may be written

$$\frac{\partial\omega}{\partial u_3}=F(\lambda\cos\omega). \tag{42}$$

Since ω is independent of u_2, λ also is independent of this parameter. Expressing the condition of integrability of equations (37) and (42), we get

$$\cos\omega\, F=\left(\frac{\partial\lambda}{\partial u_1}\cos\omega-\sin^2\omega\,\lambda\right)F'.$$

As F' cannot be zero, we write this equation in the form

$$\frac{F}{F'}=\frac{\partial\lambda}{\partial u_1}-\frac{\sin^2\omega}{\cos\omega}\lambda, \tag{43}$$

and differentiate it with respect to u_3; this gives

$$\frac{FF''}{F'^2}=-\sec^2\omega. \tag{44}$$

Now differentiate (43) with respect to u_1 and in the reduction make use of (44); this gives

$$2\frac{\partial\lambda}{\partial u_1}=\cos\omega\frac{\partial^2\lambda}{\partial u_1^2}.$$

Since ω must involve u_3, this equation can be satisfied only when λ is constant; hence equation (42) can be written

$$\frac{\partial\omega}{\partial u_3}=F(\cos\omega),$$

in which case equation (43) reduces to

$$\frac{F'}{F}=-\frac{\cos\omega}{\sin^2\omega},$$

from which it follows that in the most general case F is equal to $\sin\omega$. This result and (37) gives for ω the value

$$\tan\frac{\omega}{2}=e^{u_1+u_3},$$

which has been seen to be the value for all Weingarten systems of moulure surfaces. Hence the theorem:

The lines of equidistance in a Weingarten system are geodesic parallels when the surfaces $u_3=$ *const. are pseudospherical or moulure, and only for these systems.*

§7. *A-Systems with the Curves of Parameter u_3 Plane.*

It has been seen that when the surfaces $u_3=$ const. of an A-system are moulure surfaces the curves of parameter u_3 are plane curves. We seek now for the other A-systems having this property and shall exclude from the discussion the former systems, that is, we assume that ω involves all three parameters.

From the expression (5) for the torsion of the curves of parameter u_3 and the fundamental equations (15) and (16) it follows that the necessary and sufficient condition that these curves be plane is that ω satisfy the equation

$$\frac{\partial}{\partial u_3}\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\Big/\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)=0. \tag{45}$$

This equation can be replaced by the system

$$\left\{\begin{aligned}\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}&=\lambda\cos\alpha\frac{\partial\omega}{\partial u_3},\\ \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}&=\lambda\sin\alpha\frac{\partial\omega}{\partial u_3},\end{aligned}\right. \tag{46}$$

where α is a determined function of u_1 and u_2, and λ is an auxiliary function.

Ribaucour[*] was the first to show that if one considers a surface S of one of the families of a triple system and constructs the osculating circles of the orthogonal trajectories of S at the points of intersection with the latter, the double infinity of circles form a cyclic system. Moreover, Bianchi[*] shows that, if the orthogonal trajectories of S are plane curves, then for the triply orthogonal system of which the surfaces orthogonal to the cyclic system constitute one family the lines of curvature have the same spherical representation as for the given triple system. From this it follows that if the condition (45) is satisfied there is a corresponding A-system for which the curves of parameter u_3 are circles. We proceed to the consideration of this special system.

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[*] Bianchi, Lezioni, II, p. 487.

From (5), (14), (15), and (16) we have for the radius of principal curvature of the curves of parameter u_3 the expression

$$\frac{1}{R_3^2}=\frac{1}{\phi^2\left(\frac{\partial\omega}{\partial u_3}\right)^2}\left[\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)^2+\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)^2\right], \tag{47}$$

which by means of (46) can be written

$$\frac{1}{R_3^2}=\frac{\lambda^2}{\phi^2}.$$

Hence the necessary and sufficient condition that the curves of parameter u_3 be circles is that ϕ be of the form

$$\phi=\lambda\psi, \tag{47'}$$

where ψ is a function of u_1 and u_2. Expressing the fact that this value of ϕ must satisfy equations (21) and making use of equations (46) in the reduction, we get

$$\begin{aligned}\psi\frac{\partial\log\lambda}{\partial u_2}+\frac{\partial\psi}{\partial u_2}&=\left[F_2-\psi\int\sin\omega\frac{\partial\lambda}{\partial u_3}\,du_3\right]\sin\alpha,\\ \psi\frac{\partial\log\lambda}{\partial u_1}+\frac{\partial\psi}{\partial u_1}&=\left[F_1-\psi\int\cos\omega\frac{\partial\lambda}{\partial u_3}\,du_3\right]\cos\alpha.\end{aligned} \tag{48}$$

Differentiating with respect to u_3, we get for the equations to be satisfied by λ

$$\begin{aligned}\frac{\partial\omega}{\partial u_3}\frac{\partial^2\log\lambda}{\partial u_2\,\partial u_3}+\frac{\partial^2\omega}{\partial u_2\partial u_3}\frac{\partial\log\lambda}{\partial u_3}&=0,\\ \frac{\partial\omega}{\partial u_3}\frac{\partial^2\log\lambda}{\partial u_1\,\partial u_3}+\frac{\partial^2\omega}{\partial u_1\partial u_3}\frac{\partial\log\lambda}{\partial u_3}&=0,\end{aligned} \tag{49}$$

which are readily found to be compatible by means of equations (18). If the values of λ given by equations (46) are substituted in the above equations they are satisfied identically. Hence whenever ω satisfies (45) in addition to the fundamental equations (8), (18), and (19), it only remains to determine ψ from equations (48) and then the A-systems with the given representation and with circular curves of parameter u_3 are known. The compatibility of the system (25) of which (48) is a particular case, has been established and from the form of the latter it is seen that ψ can be determined by quadratures.

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[*] Bianchi, Lezioni, II, p. 498.

Equations (49) are satisfied identically when λ is independent of u_3. We shall consider this particular case at length. Now equations (18) reduce to

$$\begin{aligned} &\frac{\partial \omega}{\partial u_1} = \lambda \cos \alpha \sin \omega + \frac{1}{\lambda \sin \alpha} \frac{\partial}{\partial u_2} \lambda \cos \alpha, \\ &\frac{\partial \omega}{\partial u_2} = -\left(\lambda \sin \alpha \cos \omega + \frac{1}{\lambda \cos \alpha} \frac{\partial}{\partial u_1} \lambda \sin \alpha\right), \end{aligned} \tag{50}$$

and equations (19) can be put in the form

$$\begin{aligned} &\frac{\partial \omega}{\partial u_3}\left[\frac{1}{\cos \alpha} \frac{\partial \lambda}{\partial u_1} + \cos \omega \left(\lambda^2 - \frac{1}{U_3^2}\right)\right] = \frac{\sin \omega}{U_3} \frac{\partial}{\partial u_3}\left(\frac{1}{U_3}\right), \\ &\frac{\partial \omega}{\partial u_3}\left[\frac{1}{\sin \alpha} \frac{\partial \lambda}{\partial u_2} + \sin \omega \left(\lambda^2 - \frac{1}{U_3^2}\right)\right] = -\frac{\cos \omega}{U_3} \frac{\partial}{\partial u_3}\left(\frac{1}{U_3}\right). \end{aligned}$$

Multiply these equations by $\cos \omega$ and $\sin \omega$ respectively and add; again by $\sin \omega$ and $-\cos \omega$ and add; this gives

$$\begin{aligned} &\frac{\cos \omega}{\cos \alpha} \frac{\partial \lambda}{\partial u_1} + \frac{\sin \omega}{\sin \alpha} \frac{\partial \lambda}{\partial u_2} + \left(\lambda^2 - \frac{1}{U_3^2}\right) = 0, \\ &\frac{\sin \omega}{\cos \alpha} \frac{\partial \lambda}{\partial u_1} - \frac{\cos \omega}{\sin \alpha} \frac{\partial \lambda}{\partial u_2} - \frac{1}{\frac{\partial \omega}{\partial u_3}} \frac{1}{U_3} \frac{\partial}{\partial u_3}\left(\frac{1}{U_3}\right) = 0. \end{aligned} \tag{51}$$

Expressing the fact that ω, as given by (50), is a solution of (8), we are brought to the equation

$$\begin{aligned} &\frac{\sin \omega}{\cos \alpha} \frac{\partial \lambda}{\partial u_1} + \frac{\cos \omega}{\sin \alpha} \frac{\partial \lambda}{\partial u_2} + \left(\lambda^2 - \frac{1}{U_3^2}\right) \sin \omega \cos \omega \\ &\quad + \frac{\partial}{\partial u_1}\left(\frac{1}{\lambda \sin \alpha} \frac{\partial}{\partial u_2} \lambda \cos \alpha\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{\lambda \cos \alpha} \frac{\partial}{\partial u_1} \lambda \sin \alpha\right) = 0. \end{aligned} \tag{52}$$

Since λ is independent of u_3, for the expression $\lambda^2 - \frac{1}{U_3^2}$ to be zero, both λ and U_3 must be constant in which case equations (51) and (52) are satisfied. Assuming this function to be different from zero, we eliminate it from the first of (51) and (52) and get

$$\frac{\sin^3 \omega}{\cos \alpha} \frac{\partial \lambda}{\partial u_1} + \frac{\cos^3 \omega}{\sin \alpha} \frac{\partial \lambda}{\partial u_2} + \frac{\partial}{\partial u_1}\left(\frac{1}{\lambda \sin \alpha} \frac{\partial}{\partial u_2} \lambda \cos \alpha\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{\lambda \cos \alpha} \frac{\partial}{\partial u_1} \lambda \sin \alpha\right) = 0,$$

which is an impossible equation as ω cannot be independent of u_3. Hence when λ is independent of u_3, it is a constant and the corresponding A-systems are Weingarten systems. Without loss of generality equations (46) can now be written

$$\tag{53} \begin{aligned} \frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3} &= \cos\alpha\,\frac{\partial\omega}{\partial u_3}, \\ \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3} &= \sin\alpha\,\frac{\partial\omega}{\partial u_3}, \end{aligned}$$

and equations (50) reduce to

$$\tag{54} \begin{aligned} \frac{\partial\omega}{\partial u_1}+\frac{\partial\alpha}{\partial u_2} &= \cos\alpha\sin\omega, \\ \frac{\partial\omega}{\partial u_2}+\frac{\partial\alpha}{\partial u_1} &= -\sin\alpha\cos\omega. \end{aligned}$$

Now the expression for R_3 reduces to

$$R_3=\phi,$$

with ϕ a function of u_1 and u_2. Conversely for ϕ to be a function of u_1 and u_2, it is necessary (27) that

$$\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}=f_1(u_1,u_2)\frac{\partial\omega}{\partial u_3},\qquad \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}=f_2(u_1,u_2)\frac{\partial\omega}{\partial u_3},$$

where f_1 and f_2 are independent of u_3, which reduce as we have seen to $\cos\alpha$ and $\sin\alpha$ in (53). Hence the theorem:

Where ϕ is a function of u_1 and u_2 and independent of u_3, the curves of parameter u_3 are circles.

In arriving at this result we excluded the case where either u_1 or u_2 does not appear in ϕ. It was shown earlier that in this case the surfaces of two families of the system are moulure surfaces and the others are planes. To get the systems with ϕ independent of u_2 and u_3, in which case the curves of parameter u_3 are circles, it is only necessary to draw a family of circles in a plane and their orthogonal trajectories and rotate the plane about a cylinder.

It remains to consider in this connection the case where ϕ is a constant. We have seen that the surfaces $u_3=$ const. are moulure surfaces, which need not be considered in consequence of the above remarks, or pseudospherical surfaces. From (47′) it follows that in this case λ does not involve u_3, and consequently both λ and ψ are constants, so that R_3 is a constant. When equations (53) hold, the second of (34) is verified; and when the latter obtains, the curvature R_3 for the corresponding pseudospherical system is constant, so that Bianchi refers to such a system as a *system of Weingarten of constant curvature.* We have then the following theorem:

The only pseudospherical systems for which the orthogonal trajectories of the pseudospherical surfaces are circles are systems of Weingarten of constant curvature and all the circles are equal.

Now the equations (54) define the pseudospherical surfaces $u_3 = \text{const.}$ as the transforms of the pseudospherical surface whose linear element is

$$ds^2 = \cos^2\alpha \, du_1^2 + \sin^2\alpha \, du_2^2.$$

Hence the theorem:

Every Lame system of pseudospherical surfaces whose orthogonal trajectories are circles is composed of the Bianchi transforms of a pseudospherical surface.

We have shown* that all the transforms under angle $\frac{\pi}{2}$ of an A-surface S are arranged in such a manner with respect to S that the points on these transforms corresponding to a given point on S lie on the circumference of a circle whose axis is normal to S at the given point and consequently the circle cuts all the transforms orthogonally. Hence the transforms of an A-surface under angle $\frac{\pi}{2}$ form a family of Lame with circular orthogonal trajectories.

It has been shown† likewise that if an A-surface be transformed so that the tangent planes to the transforms make the same angle σ with the tangent plane to the given surface the points of contact of the former corresponding to the similar point for the latter lie on the circumference of a circle whose axis is normal to the given surface at the given point. It is evident that for this case the curves of parameter u_3 are not orthogonal to the surface, where we have denoted the transforms by $u_3 = \text{const.}$, unless σ is a right angle.

It remains to be seen whether there are families of transforms for σ a function of u_3 such that they form part of an A-system. From our previous discussion it follows that all the pseudospherical surfaces associated with these transforms are the transforms under the corresponding angles σ of the pseudospherical surface associated with the given A-surface. Moreover, the A-surfaces will not form part of an A-system unless the associated pseudospherical surfaces form such a system. It has been remarked that the points of contact of the transforms of a pseudospherical surface of whatever angle σ lie in the tangent plane to the given surface at the corresponding point. Hence for the system under consideration the curves of parameter u_3 lie in the tangent planes to the given pseudospherical surface and hence cut the tangent planes to the transforms under an angle different from a right angle. Hence we have the theorem:

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* Surfaces with the same etc. l. c. p. 138.

† l. c. p. 152.

The only family of Bäcklund transforms of an A-surface forming part of an A-system is the family for which σ is a right angle.

This result suggests the inquiry as to whether all A-systems with circular orthogonal trajectories and with ϕ a function of u_1 and u_2 are obtained in this way. For this case ω is given by (53) and (54), and equation (8) in which U_3 has been replaced by unity. Equations (54) give a transformation of angle $\alpha + \pi$ for each of the surfaces $u_3 = \text{const.}$ such that the lines of curvature on all these transforms have the same spherical images, the linear element of the sphere being

$$d\sigma^2 = \sin^2\alpha \, du_1^2 + \cos^2\alpha \, du_2^2.$$

The linear element of each of these transforms is of the form*

$$ds^2 = \left(\frac{\partial H_1}{\partial u_1} + H_2 \frac{\partial \omega}{\partial u_1} - \lambda \cos\alpha\right)^2 du_1^2 + \left(\frac{\partial H_2}{\partial u_3} - H_1 \frac{\partial \omega}{\partial u_2} - \lambda \sin\alpha\right) du_2^2,$$

where $\lambda = H_1 \cos\omega + H_2 \sin\omega$. It is evident that H_1, H_2 and ω are functions of u_3. But by means of (12) and (53) it is readily found that the coefficients in the above expression are independent of u_3, so that all the transforms coincide. Hence the theorem:

The A-systems for which ϕ is a function of u_1 and u_2 are composed of the transforms of an A-surface and their circular trajectories.

And since equations (53) are a consequence of (54) we have the converse theorem:

For all A-systems formed of the complementary transforms of an A-surface the function ϕ is independent of u_3.

§8. *A-Systems with the Curves of Parameter u_1 Plane.*

We have seen elsewhere† that the lines of curvature in both systems are plane for moulure surfaces and that there are certain A-surfaces on which the lines of curvature in one system are plane. A-Systems with a family of moulure surfaces have been considered so that we propose now a study of those A-systems for which the curves of parameter u_1 are plane. For this we must have*

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* l. c. p. 131.

† Certain Surfaces with Plane and Spherical Lines of Curvature, l. c. p. 50.

$$\frac{\partial \omega}{\partial u_2} = \mu \sin \omega , \tag{55}$$

where μ is independent of u_1, or the equivalent condition

$$\frac{\partial \omega}{\partial u_1} = \lambda \sin \omega , \tag{56}$$

where λ does not involve u_2. If these values be substituted in equation (8), the latter becomes

$$\cos \omega = \frac{\dfrac{\partial \mu}{\partial u_2} - \dfrac{\partial \lambda}{\partial u_1}}{\lambda^2 - \mu^2 - \dfrac{1}{U_3^2}} . \tag{57}$$

Eliminating ω from this equation and (55), we get

$$\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right)^2 - \left(\frac{\partial \mu}{\partial u_2}\right)^2 = \frac{\dfrac{\partial^2 \lambda}{\partial u_1^2}}{\lambda} \left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right) - \left(\frac{\partial \lambda}{\partial u_1}\right)^2 , \tag{58}$$

which upon differentiation with respect to u_2 becomes

$$\frac{\partial \mu}{\partial u_2} \left[\mu \, \frac{\partial^2 \lambda}{\partial u_1^2} \Bigg/ \frac{\partial \lambda}{\partial u_1} - 2\mu \left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right) - \frac{\partial^2 \mu}{\partial u_2^2} \right] = 0.$$

If $\dfrac{\partial \mu}{\partial u_2}$ is equal to zero, it follows from (57) that ω does not involve u_2, and consequently we should be dealing with a system of moulure surfaces. Equating to zero the expression in the parenthesis and differentiating with respect to u_1, we get

$$\frac{\partial}{\partial u_1} \left(\frac{\dfrac{\partial^2 \lambda}{\partial u_1^2}}{\dfrac{\partial \lambda}{\partial u_1}} \right) - 4 \lambda \frac{\partial \lambda}{\partial u_1} = 0.$$

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* Certain Surfaces with Plane and Spherical Lines of Curvature, l. c. p. 50.

Upon integration this becomes

$$\left(\frac{\partial\lambda}{\partial u_1}\right)^2 = \lambda^4 + 2a\lambda^2 + b,$$

where a and b are at most functions of u_3. If this value be substituted in (58), it becomes upon integration

$$(58') \qquad \left(\frac{\partial\mu}{\partial u_2}\right)^2 = \left(\mu^2 + \frac{1}{U_3^2}\right)^2 + 2a\left(\mu^2 + \frac{1}{U_3^2}\right) + b.$$

It is readily found that a and b cannot be zero at the same time.

For the second of equations (18) to be satisfied when ω is given by equations (55) and (56) it is necessary that λ be independent of u_3, hence a and b are constants in the above expression. Noting this fact, we make a similar substitution in the second of equations (19) and get

$$2\frac{\partial\omega}{\partial u_3}\frac{\partial\mu}{\partial u_2} + \sin\omega\frac{\partial}{\partial u_3}\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right) = 0.$$

If ω be eliminated from this equation and (57), the result is

$$\left[\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right)^2 + \left(\frac{\partial\mu}{\partial u_2}\right)^2 + \left(\frac{\partial\lambda}{\partial u_1}\right)^2\right]\frac{\partial}{\partial u_3}\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right)$$
$$- \left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right)\frac{\partial}{\partial u_3}\left(\frac{\partial\mu}{\partial u_2}\right)^2 = 0,$$

which equation is found to be satisfied identically when $\frac{\partial\lambda}{\partial u_1}$ and $\frac{\partial\mu}{\partial u_2}$ are replaced by their above expressions. Hence given two solutions λ and μ of the above equations in which a and b are arbitrary constants and U_3 is an arbitrary function of u_3, then ω given by (57) determines the spherical representation of a whole group of A-systems in which the curves of parameter u_1 are plane.

By means of (56) the second of equations (21) can be reduced to

$$(59) \qquad \frac{\partial\phi}{\partial u_1} = \left(F_1 - \int \cos\omega\frac{\partial\phi}{\partial u_3}\,du_3\right)\lambda.$$

If this equation and (56) be differentiated with respect to u_3 and $\cos\omega$ eliminated from the resulting equations, one gets

$$\frac{\partial\omega}{\partial u_3}\frac{\partial^2\phi}{\partial u_1\partial u_3} + \frac{\partial\phi}{\partial u_3}\frac{\partial^2\omega}{\partial u_1\partial u_3} = 0,$$

of which the general integral has the form

$$\phi = \int \frac{f(u_2, u_3)}{\frac{\partial \omega}{\partial u_3}} du_3 + \psi(u_1, u_2),$$

where f is a function independent of u_1, and ψ does not involve u_3. When this value is substituted in (59), it is found that

$$F_1 \lambda = \frac{\partial \psi}{\partial u_1}$$

and from the second of equations (20) it is found that F_2 is a function of u_2 alone. Since ϕ must satisfy the third of equations (27), one finds at once that the function $f(u_2, u_3)$ is an integral of a partial differential equation of the second order of the Laplace type. The functions ψ and F_2 are determined by the equations of condition (20); it is readily seen that the equation for the former is reducible to the form of an ordinary linear equation of the first order. Later we shall find particular A-systems of the kind which we have here considered in general.

In the A-system of which the surfaces $u_3 =$ const. are moulure surfaces the curves of parameter u_1 and of parameter u_3 are plane. We inquire whether there are any other A-systems having this property. Now ω must satisfy equations (46), (55), and (56), of which, as we have seen, the last can be written

$$\frac{1}{\sin \omega} \frac{\partial \omega}{\partial u_1} = U_1,$$

where U_1 is a function of u_1 alone. If this value of $\frac{\partial \omega}{\partial u_1}$ be substituted in the first of (46), one gets

$$\lambda \cos \alpha = U_1,$$

from which it follows that λ is independent of u_3 and consequently a constant; and moreover α is a function of u_1 alone. Again, since λ is constant, the system is a Weingarten system and hence from (58′) we have that μ does not involve u_3. From the second of (46) and (55), we get

$$\mu = \lambda \sin \alpha \tan \omega,$$

which is an impossible equation, since ω must involve u_3. Hence

Moulure A-systems are the only ones in which the orthogonal trajectories in two systems are plane curves.

§9. *A-Systems with the Curves of Parameter u_3 of Constant Curvature.*

We have met with pseudospherical systems for which all the curves of parameter u_3 have constant principal curvature. We seek now all A-systems which have this property and without loss of generality can take the radius of principal curvature equal to unity.

For this case we have from (47)

$$\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)^2+\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)^2=\phi^2\left(\frac{\partial\omega}{\partial u_3}\right). \tag{60}$$

In accordance with this relation we introduce an auxiliary function α defined by

$$\begin{aligned}\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}&=\phi\cos\alpha\frac{\partial\omega}{\partial u_3},\\ \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}&=\phi\sin\alpha\frac{\partial\omega}{\partial u_3}.\end{aligned} \tag{61}$$

In consequence of these relations equations (18) can be put in the form

$$\left\{\begin{aligned}\frac{\partial\phi}{\partial u_1}&=-\phi\cot\alpha\left(\frac{\partial\alpha}{\partial u_1}+\frac{\partial\omega}{\partial u_2}\right)-\phi^2\cos\alpha\cos\omega,\\ \frac{\partial\phi}{\partial u_2}&=\phi\tan\alpha\left(\frac{\partial\alpha}{\partial u_2}+\frac{\partial\omega}{\partial u_1}\right)-\phi^2\sin\alpha\sin\omega;\end{aligned}\right. \tag{62}$$

and in place of (19) we have

$$\left\{\begin{aligned}\frac{\partial\phi}{\partial u_1}&=\phi\tan\alpha\left(\frac{\partial\alpha}{\partial u_1}+\frac{\partial\omega}{\partial u_2}\right)-\phi^2\cos\alpha\cos\omega+\frac{\frac{\partial}{\partial u_3}\left(\frac{\sin\omega}{U_3}\right)}{U_3\cos\alpha\frac{\partial\omega}{\partial u_3}},\\ \frac{\partial\phi}{\partial u_2}&=-\phi\cot\alpha\left(\frac{\partial\alpha}{\partial u_2}+\frac{\partial\omega}{\partial u_1}\right)-\phi^2\sin\alpha\sin\omega-\frac{\frac{\partial}{\partial u_3}\left(\frac{\cos\omega}{U_3}\right)}{U_3\sin\alpha\frac{\partial\omega}{\partial u_3}}.\end{aligned}\right. \tag{63}$$

Equating the above expressions for $\frac{\partial\phi}{\partial u_1}$ and $\frac{\partial\phi}{\partial u_2}$, we get

$$\left\{\begin{aligned}&\phi\left(\frac{\partial\alpha}{\partial u_1}+\frac{\partial\omega}{\partial u_2}\right)+\frac{\sin\alpha\frac{\partial}{\partial u_3}\left(\frac{\sin\omega}{U_3}\right)}{U_3\frac{\partial\omega}{\partial u_3}}=0,\\ &\phi\left(\frac{\partial\alpha}{\partial u_3}+\frac{\partial\omega}{\partial u_1}\right)+\frac{\cos\alpha\frac{\partial}{\partial u_3}\left(\frac{\cos\omega}{U_3}\right)}{U_3\frac{\partial\omega}{\partial u_3}}=0.\end{aligned}\right. \tag{64}$$

It is readily shown that these equations are consistent when the former conditions are satisfied. By means of these relations the equations (62) can be put in the form

$$\frac{\partial\phi}{\partial u_1}=\frac{\cos\alpha\,\frac{\partial}{\partial u_3}\frac{\sin\omega}{U_3}}{U_3\,\frac{\partial\omega}{\partial u_3}}-\phi^2\cos\alpha\cos\omega,$$

$$\frac{\partial\phi}{\partial u_2}=-\frac{\sin\alpha\,\frac{\partial}{\partial u_3}\frac{\cos\omega}{U_3}}{U_3\,\frac{\partial\omega}{\partial u_3}}-\phi^3\sin\alpha\sin\omega;$$

and these equations are consistent from what precedes. When these values are substituted in the first of equations (27), the resulting equation can be reduced to

$$\begin{aligned}(65)\qquad&\left(\frac{\partial}{\partial u_3}\log\frac{1}{U_3}\right)^2+2\cot 2\omega\,\frac{\partial\omega}{\partial u_3}\,\frac{\partial}{\partial u_3}\log\frac{1}{U_3}\\&+U_3^2\left[\left(\frac{1}{\cos\omega}\,\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right)^2+\left(\frac{1}{\sin\omega}\,\frac{\partial^2\omega}{\partial u_2\,\partial u_3}\right)^2-\left(\frac{\partial\omega}{\partial u_3}\right)^2\right]=0.\end{aligned}$$

Differentiate this equation with respect to u_1; the result can be reduced by means of (18) and (19) to

$$\left(\sin\omega\cos\omega\,\frac{\partial^2\omega}{\partial u_1\,\partial u_3}-\frac{\partial\omega}{\partial u_1}\,\frac{\partial\omega}{\partial u_3}\right)\frac{\partial U_3}{\partial u_3}=0.$$

In like manner differentiation with respect to u_2 leads to

$$\left(\sin\omega\cos\omega\,\frac{\partial^2\omega}{\partial u_2\,\partial u_3}+\frac{\partial\omega}{\partial u_2}\,\frac{\partial\omega}{\partial u_3}\right)\frac{\partial U_3}{\partial u_3}=0.$$

Unless U_3 is a constant, the quantities in parenthesis are zero. For the latter case we have upon integration with respect to u_3

$$\frac{\partial\omega}{\partial u_1}=a\tan^2\omega,\qquad \frac{\partial\omega}{\partial u_2}=b\cot^2\omega,$$

where a and b are independent of u_3. For these equations to be compatible, we must have

$$\frac{\partial a}{\partial u_2}\tan^4\omega+4ab\tan\omega\sec^2\omega-\frac{\partial b}{\partial u_1}=0,$$

which is impossible, for ω must depend upon u_3. Hence U_3 is a constant and (65) becomes

$$\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right)^2+\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}\right)^2-\left(\frac{\partial\omega}{\partial u_3}\right)^2=0,$$

so that ϕ is equal to unity in (60). Hence the theorem:

Pseudospherical systems of Weingarten of constant curvature are the only A-systems for which the orthogonal trajectories of the A-surfaces have constant principal curvature.

§10. *A-Systems with Spherical Lines of Parameter u_1.*

In a former paper* we have considered A-surfaces whose lines of curvature in one family are spherical. We shall apply certain of the results obtained at that time in considering A-systems for which the curves of parameter u_1 are spherical.

One must have†

$$H_1=\alpha\sin\omega+\beta\frac{\partial\omega}{\partial u_2}, \tag{66}$$

where α and β are functions of u_2 and u_3. From the first of (7) it follows that

$$H_2=-\left(\alpha\cos\omega+\frac{\partial\beta}{\partial u_2}+\frac{\sin\omega\dfrac{\partial\alpha}{\partial u_2}+\beta\dfrac{\partial^2\omega}{\partial u_2^2}}{\dfrac{\partial\omega}{\partial u_2}}\right), \tag{67}$$

and the second of (7) may be written

$$\begin{aligned}&\left(\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_2}-\sin\omega\frac{\partial^2\omega}{\partial u_1\,\partial u_2}\right)\frac{\partial\alpha}{\partial u_2}\\&\qquad+\left[\frac{\partial\omega}{\partial u_2}\frac{\partial^3\omega}{\partial u_1\,\partial u_2^2}-\frac{\partial^2\omega}{\partial u_2^2}\frac{\partial^2\omega}{\partial u_1\,\partial u_2}+\frac{\partial\omega}{\partial u_1}\left(\frac{\partial\omega}{\partial u_2}\right)^3\right]\beta=0.\end{aligned} \tag{68}$$

Differentiate this equation with respect to u_1, and eliminate $\frac{\partial\alpha}{\partial u_2}$ and β from the resulting equation and (68); this gives

$$\begin{vmatrix}\cos\omega\dfrac{\partial\omega}{\partial u_1}\dfrac{\partial\omega}{\partial u_2}-\sin\omega\dfrac{\partial^2\omega}{\partial u_1\,\partial u_2} & \dfrac{\partial\omega}{\partial u_2}\dfrac{\partial^3\omega}{\partial u_1\,\partial u_2^2}-\dfrac{\partial^2\omega}{\partial u_2^2}\dfrac{\partial^2\omega}{\partial u_1\,\partial u_2}+\dfrac{\partial\omega}{\partial u_1}\left(\dfrac{\partial\omega}{\partial u_2}\right)^3\\ \partial\Big/\partial u_1\left[\cos\omega\dfrac{\partial\omega}{\partial u_1}\dfrac{\partial\omega}{\partial u_2}-\sin\omega\dfrac{\partial^2\omega}{\partial u_1\,\partial u_2}\right] & \dfrac{\partial}{\partial u_1}\left[\dfrac{\partial\omega}{\partial u_2}\dfrac{\partial^3\omega}{\partial u_1\,\partial u_2^2}-\dfrac{\partial^2\omega}{\partial u_2^2}\dfrac{\partial^2\omega}{\partial u_1\,\partial u_2}+\dfrac{\partial\omega}{\partial u_1}\left(\dfrac{\partial\omega}{\partial u_2}\right)^3\right]\end{vmatrix}=0, \tag{69}$$

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* Certain Surfaces with Plane or Spherical Lines of Curvature, l. c. p. 60.

† Ib., p. 61.

an equation which ω must satisfy. From the first of equations (11) we get

$$H_3 = -U_3\left[\frac{\partial\alpha}{\partial u_3} + \alpha\cot\omega\frac{\partial\omega}{\partial u_3} + \frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_2}\frac{\partial\beta}{\partial u_3} + \frac{\beta}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}\right]. \tag{70}$$

If these expressions for H_1 and H_3 be substituted in equation (16), it becomes

$$\left[\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_2} - \sin\omega\frac{\partial^2\omega}{\partial u_1\,\partial u_2}\right]\frac{\partial\beta}{\partial u_3} - \left[\tan\omega\frac{\partial^2\omega}{\partial u_1\,\partial u_3} - \frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_3}\right]\alpha = 0. \tag{71}$$

Differentiate this equation with respect to u_1 and eliminate α and $\frac{\partial\beta}{\partial u_3}$; this gives the following additional condition for ω

$$\begin{aligned}\left(\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_2} - \sin\omega\frac{\partial^2\omega}{\partial u_1\,\partial u_2}\right)&\left[\frac{\partial\omega}{\partial u_1}\frac{\partial^3\omega}{\partial u_1^2\,\partial u_3} - \frac{\partial^2\omega}{\partial u_1^2}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right.\\ &\qquad\left. + \tan\omega\left(\frac{\partial\omega}{\partial u_1}\right)^2\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right]\\ = \left(\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_3} - \sin\omega\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right)&\left[\frac{\partial\omega}{\partial u_1}\frac{\partial^3\omega}{\partial u_1^2\,\partial u_3} - \frac{\partial^2\omega}{\partial u_1^2}\frac{\partial^3\omega}{\partial u_1\,\partial u_3} + \left(\frac{\partial\omega}{\partial u_1}\right)^3\frac{\partial\omega}{\partial u_3}\right].\end{aligned} \tag{72}$$

It is readily seen that if ω satisfies (69) and (72) in addition to the fundamental equations (8), (18), (19), it remains for H_1 and H_2 to satisfy (12), which by means of (68) and (71) can be brought to the form

$$\begin{aligned}\beta\sin\omega\frac{\partial\omega}{\partial u_2}\frac{\partial\Omega}{\partial u_3} - \Omega\left[\alpha\tan\omega\frac{\partial\omega}{\partial u_3} + \frac{\partial\alpha}{\partial u_2}\sin^2\omega\right.&\\ \left. + \beta\left(\sin\omega\frac{\partial^2\omega}{\partial u_2^2} + \frac{\sin^3\omega}{\cos\omega}\left(\frac{\partial\omega}{\partial u_2}\right)^2\right)\right] &= 0,\end{aligned} \tag{73}$$

where

$$\Omega = \frac{\partial^3\omega}{\partial u_1\partial u_2\partial u_3} - \cot\omega\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_2\partial u_3} + \tan\omega\frac{\partial\omega}{\partial u_2}\frac{\partial^2\omega}{\partial u_2\partial u_3}.$$

By means of (68) and a differentiation with respect to u_1, the functions α, β, and $\frac{\partial\alpha}{\partial u_2}$ can be eliminated from (73) thus giving another equation of condition for ω. When all of these conditions are satisfied equations (68) and (71) can be written

$$\frac{\partial\beta}{\partial u_3} = \alpha\phi_1, \qquad \frac{\partial\alpha}{\partial u_2} = \beta\phi_2, \tag{74}$$

where ϕ_1 and ϕ_2 are known functions, unless the lines $u_2 = \text{const.}$ are plane, and equation (73) is of the form

$$\lambda \frac{\partial \alpha}{\partial u_2} + \mu \alpha = 0,$$

with λ and μ known functions. Hence unless λ is zero, α is given by a quadrature and β directly from the second of (74), but the values so found must satisfy the first of these equations. When λ is zero, the determination of α requires the integration of a differential equation of the second order and β follows directly. Several particular cases of the foregoing problem can be solved and are of interest.

Consider first the case where

$$\frac{\partial \omega}{\partial u_1} \frac{\partial^3 \omega}{\partial u_1^2 \partial u_3} - \frac{\partial^2 \omega}{\partial u_1^2} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} + \tan \omega \left(\frac{\partial \omega}{\partial u_1} \right)^2 \frac{\partial^2 \omega}{\partial u_1 \partial u_3} = 0,$$

which by integration with respect to u_1 becomes

$$\frac{1}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} = \rho \frac{\partial \omega}{\partial u_1}, \tag{75}$$

where ρ is a function independent of u_1. If this value be substituted in the first of equations (18), one gets

$$\frac{1}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3} \cdot \frac{\partial \omega}{\partial u_1} = \frac{\partial \omega}{\partial u_1} \frac{\partial \rho}{\partial u_2} + \rho \frac{\partial^2 \omega}{\partial u_1 \partial u_2}.$$

For these values the second of equations (18) becomes

$$\frac{\partial \omega}{\partial u_1} \frac{\partial^3 \omega}{\partial u_1^2 \partial u_2} - \frac{\partial^2 \omega}{\partial u_1^2} \frac{\partial^2 \omega}{\partial u_1 \partial u_2} + \left(\frac{\partial \omega}{\partial u_1} \right)^2 \frac{\partial \omega}{\partial u_2} = 0.$$

Then equation (72) is satisfied and also (69), as has been seen before. It is readily found that for this case Ω is zero. Hence when ω is such that the ratio of $\frac{1}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3}$ and $\frac{\partial \omega}{\partial u_1}$ is independent of u_1, there is a double infinity of A-systems for which the curves of parameter u_1 are spherical. From the investigation* of pseudospherical surfaces with spherical lines of curvature it follows that for the pseudospherical systems associated with the above A-systems the curves of parameter u_1 are spherical.

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* Certain Surfaces with Plane or Spherical Lines of Curvature, l. c., p. 62.

If we have

$$\cos\omega \frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_2} - \sin\omega \frac{\partial^2\omega}{\partial u_1 \partial u_2} = 0, \tag{76}$$

that is, if the lines $u_2 = \text{const.}$ on the surfaces $u_3 = \text{const.}$ are plane, and this equation is differentiated with respect to u_3, it can be reduced by means of (18) to

$$\cos\omega \frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_3} - \sin\omega \frac{\partial^2\omega}{\partial u_1 \partial u_3} = 0. \tag{77}$$

By means of (76) it can be shown that this equation satisfies the condition (75). Moreover, the above equations can be written

$$\frac{\partial}{\partial u_3}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_1}\right) = 0, \qquad \frac{\partial}{\partial u_2}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_1}\right) = 0,$$

from which it follows that

$$\tan\frac{\omega}{2} = F(u_1) + \psi(u_2, u_3), \tag{78}$$

where F involves only u_1 and ψ is independent of this parameter. From this we have

$$\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_2} = \frac{\partial\psi}{\partial u_2}, \tag{79}$$

and recalling the discussion for curves $u_2 = \text{const.}$ plane, we remark that ψ must satisfy the condition

$$\left(\frac{\partial^2\psi}{\partial u_2^2}\right)^2 = \left[\left(\frac{\partial\psi}{\partial u_2}\right)^2 + \frac{1}{U_3^2}\right]^2 + \left[\left(\frac{\partial\psi}{\partial u_2}\right)^2 + \frac{1}{U_3^2}\right]a + b,$$

where a and b are arbitrary constants which cannot be zero simultaneously. Again we have

$$\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_1} = F',$$

so that for the determination of F we have the equation

$$F''^2 = F'^4 + 2aF'^2 + b.$$

Enneper* has given the first integral of this equation in the form

$$F' = \sqrt{a - A\cosh 2F + B\sinh 2F},$$

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* Göttingen, Nachrichten, 1868, p. 258.

where A and B are constants bound by the relation

$$B^2 - A^2 = b + 3a^2.$$

For this value of ω equations (68), (71), (73) vanish identically so that α and β are arbitrary functions of u_2 and u_3. From (79) it follows that the expression for H_1 can be reduced to

$$H_1 = \sin\omega\left(\alpha + \beta\frac{\partial\psi}{\partial u_2}\right),$$

consequently the linear element of space for these systems can be written in the form

$$ds^2 = a^2\sin^2\omega\, du_1^2 + \left(a\cos\omega + \frac{\sin\omega}{\dfrac{\partial\omega}{\partial u_2}}\frac{\partial a}{\partial u_2}\right)^2 du_2^2 + U_3^2\left[\frac{\partial a}{\partial u_3} + a\cot\omega\frac{\partial\omega}{\partial u_3}\right]^2 du_3^2.$$

Hence we have the theorem:

A function ω given by (78) *determines the spherical representation of an infinity of A-systems for which the curves of parameter u_1 are circles.*

§11. *A-Systems with the Surfaces $u_2 = \text{const.}$ Spheres.*

From the definition of the function α in (66)* it follows that the vanishing of α is the necessary and sufficient condition that the spheres, upon which the curves of parameter u_1 lie, cut the surface orthogonally. From the theorem of Darboux† that, when the surfaces in two families cut one another orthogonally in the lines of curvature of both they form part of a triply orthogonal system, it follows that the vanishing of α is the necessary and sufficient condition that the surfaces $u_2 = \text{const.}$ are spheres.

For α to be zero it is necessary in (68) that

$$\frac{\partial\omega}{\partial u_2}\frac{\partial^3\omega}{\partial u_1\partial u_2^2} - \frac{\partial^2\omega}{\partial u_2^2}\frac{\partial^2\omega}{\partial u_1\partial u_2} + \frac{\partial\omega}{\partial u_1}\left(\frac{\partial\omega}{\partial u_2}\right) = 0 \tag{80}$$

and from (71) it is seen that either β must be independent of u_3 or the lines $u_2 = \text{const.}$ be plane on the surfaces $u_3 = \text{const.}$ Let β be independent of u_3; the only other equation of condition to be satisfied is (12) which reduces to

$$\frac{\partial\omega}{\partial u_2}\frac{\partial^3\omega}{\partial u_2^2\partial u_3} - \frac{\partial^2\omega}{\partial u_2^2}\frac{\partial^2\omega}{\partial u_2\partial u_3} - \cot\omega\frac{\partial^2\omega}{\partial u_2\partial u_3} = 0,$$

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* l. c., p. 69.

† Bianchi, Lezioni, p. 454.

and an integration with respect to u_3 gives

$$\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}=\sigma\frac{\partial\omega}{\partial u_2}, \tag{81}$$

where σ is a function independent of u_2. Proceeding with (81) in a manner analogous to the treatment of equation (75), we are brought to (80). We have then the theorem:

When the spherical representation of an A-system is such that the ratio of $\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}$ *and* $\frac{\partial\omega}{\partial u_2}$ *is independent of* u_2, *there exist an infinity of A-systems with the same spherical representation and the surfaces* $u_2=$ *const. are spheres.*

The linear element of these systems is reducible to

$$ds^2=\left(U_2\frac{\partial\omega}{\partial u_2}\right)^2du_1^2+\left(\frac{U_2\frac{\partial^2\omega}{\partial u_2^2}}{\frac{\partial\omega}{\partial u_2}}+U_2'\right)^2du_2^2+U_1^2U_3^2\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)^2du_3^2, \tag{82}$$

where U_2 is an arbitrary function of u_2 alone and the prime denotes differentiation. From the expression for the principal radius ρ_{21} in (4) it is found that ρ_{21} is equal to U_2. Consequently when U_2 is given a constant value all the spheres have the same radius.

It can be shown that the condition that the lines of parameter u_2 be plane, namely $\frac{\partial}{\partial u_2}\left(\frac{1}{\cos\omega}\frac{\partial\omega}{\partial u_1}\right)=0$, satisfies (81) in a manner similar to that followed in the discussion of equation (75), and the form of ω can be determined in an analogous manner. Hence:

Among all the A-systems with the lines of parameter u_2 *plane and having the same spherical representation, there is an infinity for which the surfaces* $u_3=$ *const. are spheres.*

The linear element of these systems is reducible to (82). For this case

$$\frac{1}{\cos\omega}\frac{\partial\omega}{\partial u_2}=f(u_2),$$

where f is a function of u_2 alone. When U_2 in (82) is replaced by the reciprocal of f, the linear element becomes

$$ds^2=\cos^2\omega\,du_1^2+\sin^2\omega\,du_2^2+U_3^2\left(\frac{\partial\omega}{\partial u_3}\right)du_3^2,$$

that is, the corresponding system is pseudospherical. Bianchi has considered* these systems for U_3 a constant and has shown that the surfaces $u_3 =$ const. are surfaces of Enneper with the same axis. When in particular U_3' is a constant, these surfaces $u_3 =$ const. are surfaces of Dini; they have been discussed by Bianchi.†

Having thus considered the case where β is independent of u_3 in (71), it remains to discuss the case where the lines of parameter u_1 are plane. Since ρ_{21} must be a function of u_2 alone, it follows for this case also that β is a function of u_2 alone. Hence:

Among the A-systems having the same spherical representation and with the lines of parameter u_1 circular, there is an infinity for which the surfaces $u_2 =$ const. are spheres.

§12. *A-Systems with One Family of Surfaces of Bianchi.*

We have shown‡ that among a group of A-surfaces with the same spherical representation there are certain surfaces which have the property that the spheres described on the segments of the normals comprised between the centers of curvature as diameters cut a fixed sphere in great circles, or orthogonally, or all pass through the center; these we have called the *surfaces of Bianchi* of the *elliptic, hyperbolic,* and *parabolic* types respectively. We seek for A-systems in which all the surfaces in one family are surfaces of Bianchi of one of the three types.

Consider a pseudospherical surface S with the linear element

$$ds^2 = \cos^2\omega \, du_1^2 + \sin^2\omega \, du_2^2, \tag{83}$$

where ω is a particular solution of the last of equations (6). By X_1, Y_1, Z_1; X_2, Y_2, Z_2 we denote the direction-cosines of the directions of the lines of curvature $u_2 =$ const. and $u_1 =$ const. on this surface, and we define two functions α and β by the equations

$$\left\{\begin{aligned} \frac{\partial \alpha}{\partial u_1} &= \cos\theta\cos\omega\ , & \frac{\partial \alpha}{\partial u_2} &= \sin\theta\sin\omega, \\ \frac{\partial \beta}{\partial u_1} &= e^{\alpha}\sin\theta\cos\omega; & \frac{\partial \beta}{\partial u_2} &= -e^{\alpha}\cos\theta\sin\omega, \end{aligned}\right. \tag{84}$$

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* Annali, vol. 14, p. 115.

† Annali, vol. 13, p. 52

‡ Surfaces with the same etc., l. c. p. 130.

when θ is a solution of the system (34′). Following the method of Bianchi we found that a surface of the parabolic type, Σ, with the spherical representation

$$d\sigma^2 = \cos^2\theta\, du_1^2 + \sin^2\theta\, du_2^2$$

is given by

$$\xi = (e^\alpha \cos\theta + \beta \sin\theta) X_1 + (e^\alpha \sin\theta - \beta\cos\theta) X_2 \tag{85}$$

and analogous expressions for η and ζ. The linear element is

$$ds^2 = (e^\alpha \cos\theta + \beta \sin\theta) du_1^2 + (e^\alpha \sin\theta - \beta\cos\theta) du_2^2. \tag{86}$$

Each solution θ of equations (34′) gives a corresponding surface Σ and now we inquire under what conditions all of these surfaces constitute a family of Lamé. It is clear that the pseudospherical surfaces having the same spherical representation as these surfaces are the complimentary transforms of the surface S by means of the function θ. Consequently this function satisfies the equations of condition (8), (18), and (19), and since H_1 and H_2 in (86) satisfy equations (7), the only remaining condition is (12), which in consequence of (84) can be reduced to

$$e^\alpha \frac{\partial\alpha}{\partial u_3} + \beta \frac{\partial\theta}{\partial u_3} = 0. \tag{87}$$

From this it is seen that, when α is known, β is given directly and it is readily shown that this value satisfies (84). Since ω in (83) is independent of u_3, equations (34′) become upon differentiation with respect to u_3,

$$\begin{aligned} \frac{1}{\cos\theta} \frac{\partial^2\theta}{\partial u_1 \partial u_3} &= \cos\omega \frac{\partial\theta}{\partial u_3}, \\ \frac{1}{\sin\theta} \frac{\partial^2\theta}{\partial u_2 \partial u_3} &= \sin\omega \frac{\partial\theta}{\partial u_3}, \end{aligned} \tag{88}$$

so that the first of equations (84) may be replaced by

$$\frac{\partial\alpha}{\partial u_1} = \frac{\dfrac{\partial^2\theta}{\partial u_1\partial u_3}}{\dfrac{\partial\theta}{\partial u_3}}, \quad \frac{\partial\alpha}{\partial u_2} = \frac{\dfrac{\partial^2\theta}{\partial u_2\partial u_3}}{\dfrac{\partial\theta}{\partial u_3}},$$

from which it follows that

$$e^\alpha = U_3 \frac{\partial\theta}{\partial u_3},$$

when U_3 is an arbitrary function of u_3. Then from (87) we have

$$\beta = -\left[\frac{U_3 \dfrac{\partial^2\theta}{\partial u_3^2}}{\dfrac{\partial\theta}{\partial u_3}} + U_3'\right],$$

where the prime denotes differentiation. By means of (11) the coefficient H_3 is readily found, so that we get for the linear element of the system

$$(89)\quad ds^2 = (e^\alpha \cos\theta + \beta \sin\theta)^2\, du_1^2 + (e^\alpha \sin\theta - \beta\cos\theta)^2\, du_2^2 + \left(e^\alpha \frac{\partial\theta}{\partial u_3} - \frac{\partial\beta}{\partial u}\right)^2 du_3^2.$$

From (88) it follows that the lines of parameter u_3 in this system are plane curves.

We consider now a pseudospherical system of Weingarten with the linear element

$$(90)\qquad ds^2 = \cos^2\omega\, du_1^2 + \sin^2\omega\, du_2^2 + \left(\frac{\partial\omega}{\partial u_3}\right)^2 du_3^2,$$

and inquire under what conditions each of the pseudospherical surfaces admits a complimentary transform with which is associated a surface of Bianchi of the parabolic type in such a way that the latter surfaces form a family of Lame. From the results of the general discussion of Bäcklund transformations it follows that θ for the transformed system must be a solution of equations (34′) and (30) of which the latter reduces to

$$(91)\qquad \frac{\cos\theta}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3} + \frac{\sin\theta}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3} + \frac{\partial\omega}{\partial u_3} = 0.$$

We have remarked that such a solution is impossible when $F(u_3)$ in (33) is negative and subsequent developments will show that for the case under discussion $F(u_3)$ cannot vanish. Moreover, it was seen that θ so determined satisfies equations (8), (18), and (19), so that only (7) and (12) remain to be considered. But the former is satisfied by

$$H_1 = e^\alpha \cos\theta + \beta\sin\theta, \qquad H_2 = e^\alpha \sin\theta - \beta\cos\theta,$$

and then the latter reduces to (87). Differentiate the latter with respect to u_3; by means of (34′) and (84) this can be reduced to

$$(92)\qquad \left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3} + \cos\theta\frac{\partial\omega}{\partial u_3}\right)\beta - e^\alpha \sin\theta\frac{\partial\omega}{\partial u_3} = 0.$$

Eliminating β between this equation and (87), we get

$$(93) \qquad \frac{\partial \alpha}{\partial u_3}\left[\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3} + \cos\theta\frac{\partial\omega}{\partial u_3}\right] + \sin\theta\frac{\partial\omega}{\partial u_3}\frac{\partial\theta}{\partial u_3} = 0.$$

If the expression in parenthesis were zero, equation (91) would reduce to

$$\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3} - \sin\theta\frac{\partial\omega}{\partial u_3} = 0,$$

and consequently the original pseudospherical system would be one of constant curvature, which has been excluded. Hence α is given by (84) and (93) to within an additive constant and β directly by (87). All the conditions of integrability are found to be satisfied and consequently we have a family of surfaces of Bianchi of the parabolic type with the linear element (89) where α and β have the above values. The above results were obtained by Bianchi[*] by a straightforward treatment of triply orthogonal systems, but we have seen fit to deduce them in a different manner as an example of the general theory of transformations of A-systems and especially as they furnish the basis of an investigation of the possible existence of systems of surfaces of Bianchi of the elliptic and hyperbolic types.

In order to make clear our procedure it is necessary to recall a theorem of Bianchi:[†] *If one takes any surface Σ of the parabolic type and a sphere S_0 with center at the origin, the circles normal to Σ and which cut the sphere in diametrically opposite points, or orthogonally, admit an infinity of normal surfaces which are in every case surfaces of the parabolic type and the axes of these circles are cut orthogonally by a family of surfaces of the elliptic type in the former case and of the hyperbolic type in the latter case; when S_0 reduces to a point the surfaces normal to the axes are of the parabolic type.* Hence with every surface of the parabolic type we can adjoin a surface of the elliptic, hyperbolic or parabolic type after the above manner.

Let the given parabolic surface Σ be defined by (85). Then the rectangular coordinates of the adjoined surface are[‡]

$$(94) \quad x = \left[\tfrac{1}{2}\left\{e^{\alpha} - (\beta^2 + \varkappa)e^{-\alpha}\right\}\cos\theta + \beta\sin\theta\right]X_1 + \left[\tfrac{1}{2}\left\{e^{\alpha} - (\beta^2 + \varkappa)e^{-\alpha}\right\}\sin\theta - \beta\cos\theta\right]X_2 - t\,X_3,$$

---

[*] Annali, ser. 3, vol. 24, p. 376.

[†] Annali, vol. 24, p. 367.

[‡] Annali, ser. 3, vol. 24, p. 368.

and similarly for y and z, where t is given by

$$
(95)\qquad
\begin{aligned}
\frac{\partial t}{\partial u_1} &= [\tfrac{1}{2}\{e^{\alpha}-(\beta^2+\varkappa)e^{-\alpha}\}\cos\theta+\beta\sin\theta]\sin\omega,\\
\frac{\partial t}{\partial u_2} &= [\tfrac{1}{2}\{e^{\alpha}-(\beta^2+\varkappa)e^{-\alpha}\}\sin\theta-\beta\cos\theta]\cos\omega,
\end{aligned}
$$

where $\varkappa$ being constant which is positive or negative according as the surface is of the elliptic or hyberbolic type, and vanishes when it is of the parabolic type, and ω is given by (34′). For this case we have

$$
(96)\qquad
\begin{aligned}
H_1 &= \tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}\cos\omega+t\sin\omega,\\
H_2 &= \tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}\sin\omega-t\cos\omega.
\end{aligned}
$$

We consider now the parabolic system whose linear element is given by (89), the function α being determined to within an additive constant by (84) and (93) and β directly by (87), and determine the conditions under which the surfaces associated with these parabolic surfaces in the manner mentioned above form a family of Lame. These adjoined surfaces defined by (94) have the same spherical representation as the pseudospherical surfaces (90) and consequently ω satisfies all of the equations (8), (18), and (19). The expressions for H_1 and H_2 given by (96) satisfy equations (7) and the remaining condition (12) can be reduced to

$$
(97)\qquad \tfrac{1}{2}\{e^{\alpha}-(\beta^2+\varkappa)e^{-\alpha}\}\frac{\partial\alpha}{\partial u_3}+\beta e^{-\alpha}\frac{\partial\beta}{\partial u_3}+e^{-\alpha}\frac{\partial\varkappa}{\partial u_3}+t\frac{\partial\omega}{\partial u_3}=0.
$$

When the value for t obtained from this equation is substituted in (95), the latter become in consequence of (87), (92), and (96)

$$
\begin{aligned}
\left(\cos\theta\frac{\partial\omega}{\partial u_3}+\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)\frac{\partial\varkappa}{\partial u_3} &= 0,\\
\left(\sin\theta\frac{\partial\omega}{\partial u_3}+\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)\frac{\partial\varkappa}{\partial u_3} &= 0.
\end{aligned}
$$

The quantities in the parentheses cannot vanish unless the original pseudospherical system has constant curvature, which is contrary to hypothesis. Hence $\varkappa$ is a constant, and when α and β are determined in the manner remarked above there is a system with the linear element

$$
(98)\qquad
\begin{aligned}
ds^2 = {} & [\tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}\cos\omega+t\sin\omega]^2\,du_1^2\\
&+[\tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}\sin\omega-t\cos\omega]^2\,du_2^2\\
&+\left[\tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}-\frac{\partial t}{\partial u_3}\right]^2 du_3^2,
\end{aligned}
$$

where t is given directly by (97) and all the surfaces of the family are of the elleptic, parabolic, or hyperbolic types according as $\varkappa$ is positive, zero, or negative.

In our previous discussion of surfaces of Bianchi it was shown that the complementary transform of the surface, defined by (94), by means of the function θ appearing in these expressions leads to the surface (85) where α, β, θ are the same as in (94). It follows therefore, that the parabolic system with the linear element (89) is one of the two A-systems which are the complimentary transforms of the system (98). The other transformed system also is parabolic, as we have shown in considering individual surfaces, and the linear elements of the transforms are given by

$$\begin{aligned} ds^2 = &\left[\tfrac{1}{2}\left\{e^{\alpha} + (\beta^2 + x)e^{-\alpha}\right\}\cos\theta + \beta\sin\theta + \tfrac{1}{2}\left\{e^{\alpha} + (\beta^2 + x)e^{-\alpha}\right\}\cos\theta'\right] du_1^2 \\ &+ \left[\tfrac{1}{2}\left\{e^{\alpha} - (\beta^2 + x)e^{-\alpha}\right\}\sin\theta - \beta\cos\theta + \tfrac{1}{2}\left\{e^{\alpha} + (\beta^2 + x)e^{-\alpha}\right\}\sin\theta'\right] du_2^2, \end{aligned}$$

where θ' denotes the second solution of (91).

PRINCETON UNIVERSITY.

Concerning a Certain Type of Continued Fractions Depending on a Variable Parameter.

By Thomas E. McKinney.

Part I.

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Introduction.

Every system of continued fractions is based on a defining equation related to a partitioned field with each part of which and as its representative is associated a particular number.

The field may be any provided it be precisely defined, *i. e.* it shall be possible to determine by means of a finite number of operations if any given number belongs to the field. The manner of partitioning is subject to the single limitation that every number of the field shall appear in one and only one part. The representative number associated with any part may be any number of the field; the relation, however, of representative number and corresponding part is unique. The defining equation relates any number of the field and its representative number uniquely. Otherwise expressed, the given number and its representative in conjunction with the defining equation serves to define without ambiguity a third number of the field.

In the process of developing a number into a continued fraction it is conceivable that the parts and their corresponding representatives and the defining equations, any or all, may change. It is sufficient if at every stage of this process these elements are determined without ambiguity and in accordance with the restrictions imposed in the preceding paragraph.

Every system of continued fractions thus defined assigns to every number of the field a definite, determinate continued fraction. The effectiveness of a system so defined depends when the field is given largely on the particular choice made of the parts, representative numbers and defining equations with respect to the ease with which the operations involved may be carried on and with respect to the simplicity of form in which the results may be stated.

A satisfactory and complete characterization of the defining equation, the field, the parts, and the representative numbers is the first fundamental problem in every system of continued fractions. A second fundamental problem arises in the attempt to define the numbers of the field in terms of the representative numbers, *i. e.* to assign to every number of the field a definite continued fraction. It is, perhaps, loosely analogous to a fundamental problem in function theory, viz., to define the numbers of a system in terms of numbers of an assigned character.

An important problem in the theory of quadratic forms is related to the continued fractions arising when in a continued fraction in a given system the partial denominators, barring the first, are taken in reverse order.

More fundamental is the grouping of the numbers of the field with respect to the properties of their developments; as, with respect to ultimate identity as in equivalence, with respect to periodicity, or, more generally, with respect to the properties which serve to characterize the numbers of the field as algebraic or transcendental.

The system of continued fractions studied in this paper refers to the field of real numbers. This field is partitioned by means of a parameter λ, the parts being termed intervals. The role played by this parameter suggests the name λ-developments for the continued fractions of this system. This parameter not only affords the means of giving a wide scope to the investigation, but also throws light on certain fundamental phenomena which appear only in a subdued form in the systems hitherto studied so that their real significance has not been generally remarked.

The element of symmetry has been introduced by a suitable disposition of the intervals in the positive and negative portions of the field, — so that given the λ-development $(\lambda = \lambda')$ of a positive number the like development of the corresponding negative number may be immediately derived. The representative numbers are the integers lying in the corresponding intervals. This holds without exception, though an apparent exception arises when in the process of developing a number into a continued fraction it becomes necessary to subdivide the interval. In this case the representative number of the partial or special interval is the integer lying in the corresponding normal interval, whether that integer be within or without the special interval.

With respect to genetic relationship this investigation has its origin in a paper by Hurwitz.[*] The system of λ-developments here considered is a direct generalization of Hurwitz "Die Kettenbruch-Entwicklung zweiter Art," hereafter referred to as H_{II}. Save the lack of symmetry in the interval system "Die Kettenbruch-Entwicklung erster Art," or H_I, is to be classed with H_{II}. The plan followed is substantially that of Hurwitz. His methods have been employed and his results incorporated whenever they served the purposes of this paper. Acknowledgment is made in foot notes. But the nature of the indebtedness is such that a precise formulation and acknowledgment in all cases has not seemed practicable.

For this system the λ-development $(\lambda = \lambda')$ of λ' is fundamental. It is determined directly by means of the interval system and the defining equation. On this λ-development taken as known is based the characterization of the λ-development $(\lambda = \lambda')$ of every other real number. A study of the λ-developments $(\lambda = \lambda')$ of certain simple functions of λ' is made in §2 preliminary to the treatment of the general case of any number of the field.

In §3 a finite λ-development $(\lambda = \lambda')$ is assigned to every real number and a criterion is set up for deciding whether a given finite continued fraction equal numerically to a given number is the λ-development $(\lambda = \lambda')$ of that number.

In the characterization of infinite λ-developments $(\lambda = \lambda')$ it has been necessary (§4) to study the question of the convergence of these developments. It is shown that every infinite λ-development $(\lambda = \lambda')$ is convergent and that with exceptions completely specified every infinite λ-sequence $(\lambda = \lambda')$ defines a number of which it is the λ-development $(\lambda = \lambda')$.

---

[*] Hurwitz: Ueber eine Besondere Art der Kettenbruch-Entwicklung Reeller Grössen, Acta Mathematica, Vol. 12.

Out of the study of the denominators of successive convergents of H_I emerged H_{II} as the reverse development to H_I. With an exception specified by Hurwitz the relation is reciprocal. In the general case of reverse λ-developments ($\lambda = \lambda'$) aside from the instances remarked by Hurwitz there is but one other in the continuous infinity of systems here studied and this with an exception—for which reversion in all its generality holds true.

A comparison is made in §6 of several different systems dependent on a parameter λ, so that the results of one system may be transferred in certain cases to another system without the labor of an investigation *ab initio.*

In §7 the relation of λ-developments ($\lambda = \lambda$) of numbers properly equivalent is investigated. The result is to divide equivalent numbers into two classes: 1.° Equivalent numbers whose λ-developments ($\lambda = \lambda'$) are ultimately alike; 2.° Equivalent numbers whose λ-developments ($\lambda = \lambda'$) are not ultimately alike, but are so related that the λ-developments ($\lambda = \lambda'$) of the one from elements of a certain rank on may be transformed into a corresponding position of the λ-development ($\lambda = \lambda'$) of the other by a very simple rule. The obvious extension of these results to improper equivalence closes the section.

§1. *Notation and Defining Equations.*

To explain the notation* of this paper let f and g be any real numbers $f < g$. Then the expression

$$x = f \dots g \tag{1}$$

is equivalent to

$$f \leqq x \leqq g.$$

When, however, the greater number comes first, as

$$x = g \dots f, \tag{2}$$

the interpretation is

$$x \geqq g, \text{ or } x \leqq f.$$

We say that f and g determine an interval and, indeed, are its limits. To indicate that the relation of equality does not exist between x and a limit number f, or g, *i. e.* to exclude this number from the interval it is inclosed in parentheses. Thus, the expression, or as we may sometimes say, the interval equations

---

* Hurwitz, *loc. cit.*

$$x = (f) \cdots \; g,$$
$$x = f \; \cdots (g), \qquad f < g,$$

are equivalent, respectively, to

$$f < x \leqq g,$$
$$f \leqq x < g.$$

From the definition of the symbolic equation (1) it follows that when k is any real number

$$x + k = f + k \cdots g + k \tag{3}$$
$$kx = kf \; \cdots \; kg, \quad k > 0, \tag{4}$$
$$kx = kg \; \cdots \; kf, \quad k < 0, \tag{5}$$
$$\frac{k}{x} = \frac{k}{g} \; \cdots \; \frac{k}{g}, \quad k > 0, \tag{6}$$
$$\frac{k}{x} = \frac{k}{f} \; \cdots \; \frac{k}{g}, \quad k < 0. \tag{7}$$

From equations (4), (5), (6), (7), are derived the following

$$x = \; f \cdots g, \tag{8}$$
$$-x = -g \cdots -f, \tag{9}$$
$$\frac{1}{x} = \frac{1}{g} \cdots \frac{1}{f}, \tag{10}$$
$$-\frac{1}{x} = -\frac{1}{f} \cdots -\frac{1}{g}. \tag{11}$$

An analogous system of equations follows from the definition of equation (2). When

$$x = f \cdots g \tag{12}$$

an interval not containing zero, and

$$y = h \cdots l, \tag{13}$$

an interval not containing infinity then for

$$|f| + |g| > |h| + |l|$$
$$x + y = f + h \cdots g + l. \tag{15}$$

2.° Adopting the usual representation of real numbers by points on a right line, in fact, a closed line passing through infinity so that $+\infty$ and $-\infty$ are represented by the same point, we may interpret equations (1) and (2) geometrically. The positive direction on the line is that of increasing numbers, in accordance with the usual convention, from left to right. Equation (1), or (2) now defines that interval which lies on the positive side of the first limit number, *i. e.* for equation (1) to the positive side of f and for equation (2) to the positive side of g.

In the immediate sequel brevity of statement will require the Kronecker symbol sgn. x. This symbol is defined by the equation

$$\operatorname{sgn.} x = \frac{x}{|x|} = \pm 1.$$

where x is real and not zero. When occasion requires a value will be assigned arbitrarily to sgn. x, $x = 0$.

3.° It will facilitate the treatment of the general case to have a notation which changes readily the order of the limit numbers. To do this we inclose the interval limits in parentheses with the proper sign prefixed. Thus:

$$\operatorname{sgn.} x \{a \dots . b\} = a \dots . b$$

when sgn. $x = +1$, and

$$\operatorname{sgn.} x \{a \dots . b\} = -b \dots . -a$$

when sgn. $x = -1$.

We shall have occasion to refer to the interval limits a and b. a will be called the left interval limit and b the right interval limit.

The notation of this paragraph will be employed chiefly in section 3.

4.° Now divide the series of real numbers into the intervals

$$\begin{aligned} \dots .;\ (-2-\lambda') \dots . -1-\lambda';\ (-1-\lambda') \dots . -\lambda';(-\lambda') \\ \dots .(+\lambda');\ \lambda' \dots .\ (1+\lambda');\ 1+\lambda' \dots .(2+\lambda');\dots ., \end{aligned} \tag{15}$$

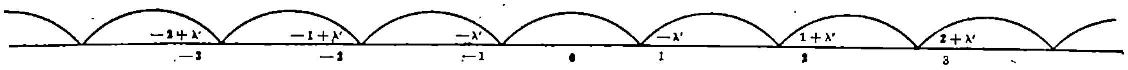
where λ' is defined by the interval equation

$$\lambda' = (0) \dots . 1. \tag{16}$$

Every real number lies in one and only one of these intervals and every interval contains one and only one integer which is called the representative number or integer of the interval. When x lies in the interval of which a is the representative number we say that a represents x, or that x corresponds to a and write

$$x \sim a.$$

The system of intervals (15) may be exhibited graphically on a right line, thus:



The interval limits are written in the intervals to which they belong.

5.° Let x_0 be any real non-zero number and on the basis of the interval system (15) form successively the equations

$$x_0 = a_0 - \frac{1}{x_1}, \quad x_1 = a_1 - \frac{1}{x_2}, \quad x_2 = a_2 - \frac{1}{x_3}, \ldots, x_{n-1} = a_{n-1} - \frac{1}{x_n}, \tag{17}$$

where

$$x_i \sim a_i, \quad i = 0, 1, 2, \ldots, n-1. \tag{18}$$

Eliminating $x_1, x_2, \ldots, x_{n-1}$ from these equations the results may be expressed as a continued fraction

$$x_0 = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 \,\ddots\, - \cfrac{1}{a_{n-1} - \cfrac{1}{x_n}}}},$$

or, according to the notation we shall adopt

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \ldots, a_{n-1}, x_n). \tag{19}$$

Continued fractions of the type (19) whose interval system (15) depends on a parameter λ we call for brevity λ-developments, and when we wish to specify the particular value of the parameter λ on which the interval system depends we shall write λ-developments $(\lambda = \lambda')$. In §6 a fuller notation will be used but there is no occasion to explain it at this point.

When it is desired to assert merely numerical equality without affirming that the second member of the equation is the λ-development $(\lambda = \lambda')$ of the first we shall write

$$x_0 = (a_0, a_1, \ldots, a_{n-1}, x_n). \tag{20}$$

When the distinction between partial and complete quotients is not important $a_0, a_1, \ldots, a_{n-1}, x_n$ will be referred to as elements* of the λ-development $(\lambda = \lambda')$ of x_0.

For a given x_0 and a given λ, say λ′, the elements $a_0, a_1, \ldots, a_{n-1}, x_n$ are uniquely determined so that the λ-development $(\lambda = \lambda')$ of x_0 is unique.

---

* Dirichlet, Zahlentheorie, §§ 23, 80.

§2. λ-*Developments* ($\lambda=\lambda'$, $0<\lambda'\leqq 1$) *of Certain Functions of* λ'.

1.° In this section as a basis* for the treatment of the more general case of any real number the λ-development ($\lambda=\lambda'$) of λ' and of certain simple functions of λ' are derived. The λ-development ($\lambda=\lambda'$) of every real number is limited in a very real, though untechnical, sense by the λ-developments ($\lambda=\lambda'$) of two of these functions; viz., $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$ and $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$. While it is possible in any particular case to derive the λ-development ($\lambda=\lambda'$) of a real number by means of the defining equations and the interval system (15), §1 directly, yet a general description of such developments involve the partial and complete quotients in $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$ and $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$. That this description may refer ultimately to the defining equations and to the interval system (15), §1 alone, to these are referred the partial and complete quotients in question. Now in the first instance since $\frac{1}{1-\lambda'}$ is a complete quotient in $\lambda\,(\lambda')_{\lambda'}$ this is done by finding $\lambda\,(\lambda')_{\lambda'}$ directly by means of the defining equations and the interval system (15), §1; in the second instance by showing how from $\lambda\,(\lambda')_{\lambda'}$ to derive $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$. In the further consideration of the subject the partial and complete quotients entering in $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$ and $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$ will be considered as known.

In the types of continued fractions hitherto studied $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$ and $\lambda\left(-\frac{1}{\lambda'}\right)$ do not play a conspicuous part. For $\lambda'=1$, which is the case corresponding most closely to ordinary continued fractions $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}=\infty$, $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}=-1$; for $\lambda'=\frac{1}{2}$, which with very slight modification gives H_I, $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}=2$, $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}=-2$; for $\lambda'=\frac{-1+\sqrt{5}}{2}$ *i. e.* for H_{II}.

$$\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}=(3,\ 3,\ 3,\dots),$$

$$\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}=(-2,-3,-3,\dots),$$

---

* The idea that $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$ and $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$ should first be derived and that their partial and complete quotients be considered as known, which effects a treatment of the general case, is due to Professor E. H. Moore.

where the real significance of these developments is veiled somewhat by their periodicity.

2.° Since $0 < \lambda' \leqq 1$, $\lambda' \sim 1$ [§1, (15)]. When $\lambda' = 1$, the $\lambda(\lambda')_{\lambda'} = 1$ and this case is disposed of completely. When $\lambda' < 1$, then ξ is determined by the equation

$$\lambda' = 1 - \frac{1}{\xi_1} = (1, \xi_1). \tag{1}$$

Hence

$$\lambda(\lambda')_{\lambda'} = (1, \xi_1), \tag{2}$$

where

$$\xi_1 = \frac{1}{1-\lambda'} = 1 + \lambda' + \lambda'^2 + \dots > 1 + \lambda'. \tag{3}$$

When ξ_1 is an integer α_1, then $\lambda(\lambda')_{\lambda'}$ cannot be extended further than in equation (2). When ξ_1 is not an integer its corresponding integer α_1 is determined by reference to the interval system (15), §1. Then ξ_2 is defined by the equation

$$\xi_1 = \alpha_1 - \frac{1}{\xi_2} = (\alpha_1, \xi_2) = \lambda(\xi_1)_{\lambda'}. \tag{4}$$

From the correspondence of ξ_1 and α_1

$$\frac{1}{\xi_2} = -\overline{1-\lambda'} \dots (\lambda'), \tag{5}$$

whence

$$\xi_2 = \frac{1}{1-\lambda'} \dots \left(-\frac{1}{\lambda'}\right). \tag{6}$$

Now by equations (2) and (4)

$$\lambda(\lambda')_{\lambda'} = (1, \alpha_1, \xi_2). \tag{7}$$

When ξ_2 is an integer α_2, the process of development terminates with ξ_2. When ξ_2 is not an integer then its corresponding integer is determined directly by reference to the interval system (15), §1. ξ_3 is then defined by the equation

$$\xi_2 = \alpha_2 - \frac{1}{\xi_3}. \tag{8}$$

For $\xi_2 > 0$,

$$\frac{1}{\xi_3} = -\overline{1-\lambda'} \dots \lambda', \tag{9}$$

and

$$\xi_3 = \frac{1}{1-\lambda'} \dots \left(-\frac{1}{\lambda'}\right); \tag{10}$$

for $\xi_2 < 0$,

$$\frac{1}{\xi_3} = (-\lambda') \ldots . 1 - \lambda', \tag{11}$$

and

$$\xi_3 = \left(\frac{1}{\lambda'}\right) \ldots . - \frac{1}{1 - \lambda'}. \tag{12}$$

In the more general notation explained in §1, 3.° Equations (10) and (12) may be briefly written

$$\xi_3 = \text{sgn.}\, \xi_2 \left\{\frac{1}{1 - \lambda'} \ldots . \left(-\frac{1}{\lambda'}\right)\right\}. \tag{13}$$

Combining equations (7) and (8)

$$\lambda\,(\lambda')_{\lambda'} = (1,\ \alpha_1,\ \alpha_2,\ \xi_3). \tag{14}$$

Again, when ξ_3 is an integer the process of development necessarily terminates with ξ_3. When ξ_3 is not an integer it may be replaced by

$$\alpha_3 - \frac{1}{\xi_4}$$

where $\xi_3 \sim \alpha_3$ and

$$\xi_4 = \text{sgn.}\, \xi_3 \left\{\frac{1}{1 - \lambda'} \ldots . \left(-\frac{1}{\lambda'}\right)\right\} = \text{sgn.}\, \alpha_3 \left\{\frac{1}{1 - \lambda'} \ldots . \left(-\frac{1}{\lambda'}\right)\right\}. \tag{15}$$

Equation (14) now becomes

$$\lambda\,(\lambda')_{\lambda'} = (1,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \xi_4). \tag{16}$$

This process of successive reduction may be continued as long as the last complete quotient found is not an integer. The result stated generally is

$$\lambda\,(\lambda')_{\lambda'} = (1,\ \alpha_1,\ \alpha_2, \ldots,\ \alpha_{i-1},\ \xi_i), \tag{17}$$

when $\xi_t \sim \alpha_t$, $\xi_t - \alpha_t = -\dfrac{1}{\xi_{t+1}}$ and

$$\xi_{t+1} = \text{sgn.}\, \alpha_t \left\{\frac{1}{1 - \lambda'} \ldots . \left(-\frac{1}{\lambda'}\right)\right\},\quad t = 0,\ 1, \ldots,\ i - 1, \tag{18}$$

i a positive integer equal to or greater than one.

By equation (18)

$$|\xi_{t+1}| > 1$$

and, hence,

$$|\alpha_{t+1}| \geqq 1, \quad t = 0,\ 1, \ldots \quad i - 1.$$

When ξ_i for any value of i, say $i = k$, is identical with its corresponding integer, say α_k, we shall uniformly write the λ-development ($\lambda = \lambda'$) of λ' thus:

$$\lambda\,(\lambda')_{\lambda'} = (1,\ \alpha_1,\ \alpha_2, \ldots,\ \alpha_k). \tag{19}$$

Applying the process just employed in the case of λ' to ξ_t we have

$$\lambda(\xi_t)_{\lambda'} = (\alpha_t, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i),\ t = 0, 1, 2, \dots i-1, \tag{20}$$

where, for $t = 0$, $\alpha_t = 1$, and $\xi_t = \lambda'$.

3.° To derive $\lambda(-\lambda')_{\lambda'}$ from $\lambda(\lambda')_{\lambda'}$ we appeal to the symmetry of the interval system (15), §1 with respect to zero. We have only to change the sign of all elements entering equations (1)–(17) to make the investigation apply to the present case. Obviously, this involves no change in the form of equations (13) and (18). Hence

$$\lambda(-\lambda')_{\lambda'} = (-1, -\alpha_1, \dots, -\alpha_{i-1} - \xi_i), \tag{21}$$

where $\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \xi_i$ have the values assigned in equation (17). In like manner it follows that

$$\lambda(-\xi_t)_{\lambda'} = (-\alpha_t, -\alpha_{t+1}, \dots, -\alpha_{i-1}, -\xi_i),\ t = 0, 1, \dots, i-1. \tag{22}$$

4.° When $\xi_t > 0$, $0 \leqq t < i$, and m a positive integer, $m < \xi_t$, then, since $\xi_t - m$ corresponds to $\alpha_t - m$

$$\lambda(\xi_t - m)_{\lambda'} = (\alpha_t - m, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i), \tag{23}$$

and for a like reason when m is any positive integer

$$\lambda(\xi_t + m)_{\lambda'} = (\alpha_t + m, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i). \tag{24}$$

When $\xi_t < 0$, $0 \leqq t < i$, we have, corresponding to equations (23) and (24), respectively, the following equations:

$$\lambda(\xi_t + m)_{\lambda'} = (\alpha_t + m, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i), \tag{25}$$

m an integer, $0 \leqq m < |\xi_t|$;

$$\lambda(\xi_t - m)_{\lambda'} = (\alpha_t - m, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i), \tag{26}$$

m any positive integer.

5.° To derive $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$ from $\lambda(\lambda')_{\lambda'}$ grant that

$$\begin{aligned}\lambda(\lambda')_{\lambda'} &= (1, 2_1, 2_2, \dots, 2_h, \xi_{h+1}) \\ &= (1, 2_1, 2_2, \dots, 2_h, \alpha_{h+1}, \dots, \alpha_{i-1}, \xi_i),\end{aligned} \tag{27}$$

where h is some positive integer and $\alpha_{h+1} \neq 2$.

By equation (27)

$$\lambda' = \frac{\xi_{h+1} - 1}{(h+1)\xi_{h+1} - h} \tag{28}$$

and

$$\frac{1}{\lambda'} = h + 1 + \frac{1}{\xi_{h+1} - 1}. \tag{29}$$

Equation (27) is a special case of the following: Let

$$x_0 = (a_0, \gamma_1, \gamma_2, \cdots, \gamma_n, \xi_{n+1}),$$

where

$$\gamma_1 = \gamma_2 = \cdots\cdots = \gamma_n \equiv \gamma.$$

Then

$$x_0 = a_0 - \frac{\phi_{n-1}(\gamma)\,\xi_{n+1} - \phi_{n-2}(\gamma)}{\phi_n(\gamma)\,\xi_{n+1} - \phi_{n-1}(\gamma)},$$

where

$$\phi_k(\gamma) = \sum_{0}^{\left[\frac{k}{2}\right]} (-1)^i \binom{k-i}{i} \gamma^{k-2i},$$

$$\phi_0(\gamma) = 1 \text{ and } \phi_{-1}(\gamma) = 0.$$

$$\text{a)} \quad \lambda'^2 + \lambda' - 1 > 0.$$

In this case

$$\lambda' > \frac{-1+\sqrt{5}}{2} \equiv \tau, \tag{30}$$

and

$$\frac{1}{1-\lambda'} > 2 + \lambda'. \tag{31}$$

Consequently for the values of λ' now in question, the condition $\alpha_{h+1} \neq 2$ is always fulfilled and under the hypothesis of equation (27)

$$\xi_{h+1} = \frac{1}{1-\lambda'} \cdots\cdots \left(-\frac{1}{\lambda'}\right). \tag{32}$$

Also, it is true that

$$\xi_h \geqq \frac{1}{1-\lambda} > 2 + \lambda, \tag{33}$$

and, consequently, that $\alpha_h > 2$, which is consistent with the hypothesis in equation (27) only when $h = 0$. Hence in this case

$$\xi_{h+1} = \frac{1}{1-\lambda'} > 0, \tag{34}$$

and, consequently,

$$h + 1 < \frac{1}{\lambda'} \equiv h + 1 + \frac{1}{\xi_1 - 1} < h + 1 + \lambda'. \tag{35}$$

Therefore,

$$\frac{1}{\lambda'} \sim h + 1, \tag{36}$$

and

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = (h+1, -(\xi_{h+1} - 1)) = (h+1, -(\alpha_1 - 1), \\ -\alpha_2, \cdots, -\alpha_{i-1}, -\xi_i), \tau < \lambda' \leqq 1. \tag{37}$$

$$\text{b)}\quad 0 < \lambda' \leqq \tau.$$

Bearing in mind the hypothesis with respect to α_{h+1}, we have

$$\xi_{k+1} = 2 + \lambda' \ldots. \left(-\frac{1}{\lambda'}\right), \tag{38}$$

and consequently,

$$\frac{1}{\xi_{k+1} - 1} = \left(\frac{-\lambda'}{1+\lambda'}\right) \ldots. \frac{1}{1+\lambda'}. \tag{39}$$

Hence

$$\begin{aligned} \frac{1}{\lambda'} \equiv h + 1 + \frac{1}{\xi_{h+1} - 1} &= \left(h + 1 - \frac{\lambda'}{1+\lambda'}\right) \ldots. h' + 1 + \frac{1}{1+\lambda'} \\ &= \left(h + \frac{1}{1+\lambda'}\right) \ldots. h' + 1 + \frac{1}{1+\lambda'}. \end{aligned} \tag{40}$$

Now, for $\lambda' \leqq \tau$,

$$h + \frac{1}{1+\lambda'} \geqq h + \lambda' \tag{41}$$

and

$$h + 1 + \lambda' \leqq h + 1 + \frac{1}{1+\lambda'} < h + 2. \tag{42}$$

Hence for the values of λ' under consideration $\frac{1}{\lambda'}$ corresponds to $h+1$ or to $h+2$.

$$\text{b}_1)\quad \frac{1}{\lambda'} \sim h + 1.$$

Then

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = (h + 1, -(\xi_{k+1} - 1)) \tag{43}$$

and

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = (h + 1, -(\alpha_{k+1} - 1), -\alpha_{h+2}, \ldots, -\alpha_{i-1}, -\xi_i). \tag{44}$$

$$\text{b}_2)\quad \frac{1}{\lambda'} \sim h + 2.$$

In this case

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = \left(h + 2, \frac{\xi_{h+1} - 1}{\xi_{h+1} - 2}\right). \tag{45}$$

Since

$$h + 1 + \frac{1}{\xi_{k+1}} \geqq h + 1 + \lambda', \tag{46}$$

then

$$0 < \xi_{k+1} \leqq \frac{1}{\lambda'} + 1. \tag{47}$$

Now

$$\frac{\xi_{k+1} - 1}{\xi_{h+1} - 2} = \left(2_1, 2_2, \ldots, 2_{k-2}, \frac{\xi_{h+1} - h + 1}{\xi_{h+1} - k}\right),\ k \text{ an integer} \geqq 2. \tag{48}$$

In order that

$$\frac{\xi_{h+1}-k+1}{\xi_{h+1}-k}\equiv 1+\frac{1}{\xi_{h-1}-k}\sim 2,$$

it is necessary and sufficient that

$$\frac{1}{\xi_{h+1}-k}=\lambda' \ldots . (1+\lambda'), \tag{49}$$

or

$$\xi_{h+1}=\left(\frac{1}{1+\lambda'}+k\right) \ldots . \frac{1}{\lambda'}+k. \tag{50}$$

By equation (47) ξ_{h+1} is less than the greater limit of this interval. Again, since

$$\xi_{h+1}\geqq \alpha_{h+1}-1+\lambda', \tag{51}$$

then for $0<\lambda'\leqq\tau$, and for $k\leqq\alpha_{h+1}-2$, ξ_{h+1} is certainly greater than the lesser interval limit in equation (50). Hence by equation (48)

$$\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'}=\left(2_1,\ 2_2,\ \ldots,\ 2_{\alpha_{h+1}-4},\ 2_{\alpha_{h+1}-3},\ \frac{\xi_{h+1}-\alpha_{h+1}+2}{\xi_{h+1}-\alpha_{h+1}+1}\right). \tag{52}$$

By the equation

$$\xi_{h+1}-\alpha_{h+1}=-\frac{1}{\xi_{h+1}}$$

equation (52) becomes

$$\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'}=\left(2_1,\ 2_2,\ \ldots,\ 2_{\alpha_{h+1}-3},\ 2+\frac{1}{\xi_{h+2}-1}\right). \tag{53}$$

The further discussion of this equation presents two cases.

$$b_{2,1})\quad 2+\frac{1}{\xi_{h+2}-1}\sim 2.$$

Equation (53) now becomes

$$\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'}=(2_1,\ 2_2,\ \ldots,\ 2_{\alpha_{h+1}-3},\ 2_{\alpha_{h+1}-2},\ -(\xi_{h+2}-1)). \tag{54}$$

Combining this with equation (45)

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}=(h+2,\ 2_1,\ 2_2,\ \ldots,\ 2_{\alpha_{h+1}-3},\ 2_{\alpha_{h+1}-2},-(\alpha_{h+2}-1),\ \ldots,-\alpha_{i+1},-\xi_i). \tag{55}$$

$b_{2,2})$ $2+\dfrac{1}{\xi_{h+1}-1}$ does not correspond to 2.

Then

$$\frac{1}{1-\lambda'}\leqq\xi_{n+1}\leqq\frac{1}{\lambda'}+1,\ \alpha_{h+2}>2, \tag{56}$$

and the further reduction of the last complete quotient $2+\frac{1}{\xi_{h+1}-1}$ in equation (53) presents precisely the problem of equation (29) for $0<\lambda'\overset{=}{<}\tau$, which has been partially solved under b), b_1) and b_2). It may be included in the following:

Let $\alpha_{h+2}=\alpha_{h+3}=\ldots=\alpha_{h+h_1+1}=2\ddagger\alpha_{h+h_1+2}$, h_1 a positive integer. Then

$$2+\frac{1}{\xi_{h+2}-1}=3+\frac{1}{\xi_{h+3}-1}=\ldots=h_1+2+\frac{1}{\xi_{h+h_1+2}-1} \tag{57}$$

In this case equation (53) reduces to

$$\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'}=\left(2_1,\ 2_2,\ldots,\ \underset{h+1}{2_{a-3}},\ h_1+2+\frac{1}{\xi_{h+h_1+2}-1}\right). \tag{58}$$

It may be remarked that for $h_1=0$ this equation includes the case specified under equation (56).

Now, when

$$\xi_{h+h_1+2}-1=\left(\frac{1}{\lambda'}+1\right)\ldots\left(-\frac{1}{\lambda'}\right), \tag{59}$$

then by equation (58)

$$\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'}=(2_1,\ 2_2,\ldots,\ \underset{h+1}{2_{a-3}},\ h+2,-(\xi_{h+h_1+2}-1)), \tag{60}$$

and the further extension of this development is immediate.

When, however,

$$\xi_{h+h_1+2}=\frac{1}{1-\lambda'}+1\ \ldots\ \frac{1}{\lambda'}+1, \tag{61}$$

it is readily shown that

$$h_1+2+\frac{1}{\xi_{h+h_1+2}-1}\sim h_1+3. \tag{62}$$

In this case equation (58) becomes

$$\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'}=\left(2_1,\ 2_2,\ldots,\ \underset{h+1}{2_{a-3}},\ 3+h_1,\frac{\xi_{h+h_1+2}-1}{\xi_{h+h_1+2}-2}\right). \tag{63}$$

Applying to $\frac{\xi_{h+h_1+2}-1}{\xi_{h+h_1+2}-2}$ the process employed in the reduction of $\frac{\xi_{h+1}-1}{\xi_{h+1}-2}$ we have at once

$$\begin{aligned}\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'}=\Big(2_1,\ 2_2,\ldots,\ \underset{h+1}{2_{a-3}},\ 3+h_1,\ 2_1,\ 2_2,\ldots,&\\ \underset{h+h_1+2}{2_{a-3}},\ \frac{\xi_{h+h_1+2}-a+2}{\xi_{h+h_1+2}-a+1}\Big).&\end{aligned} \tag{64}$$

By the equation

$$\xi_{h+h_1+2} - \alpha_{h+h_1+2} = -\frac{1}{\xi_{h+h_1+3}} \tag{65}$$

equation (64) reduces to

$$\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'} = \Big(2_1,\ 2_2, \dots,\ 2_{\substack{a-3\\h+1}},\ 3+h_1,\ 2_1,\ 2_2, \dots, \ 2_{\substack{a-3\\h+h_1+2}},\ 2+\frac{1}{\xi_{h+h_1+3}-1}\Big). \tag{66}$$

Enough has been done to indicate the process by which the development in equation (66) may be carried to any desired extent. The result in general form may be written

$$\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'} = \Big(2_1,\ 2_2, \dots,\ 2_{\substack{a-3\\h+1}},\ 3+h_1,\ 2_1,\ 2_2, \dots,\ 2_{\substack{a-3\\h+h_1+2}},\ 3+h_2, \dots,\ 3+h_{s-1},\ 2_1,\ 2_2, \dots,\ 2_{\substack{a-3\\h+h_1+\dots+h_{s-1}+s}},\ 2+\frac{1}{\xi_{h+h_1+\dots+h_{s-1}+s+1}-1}\Big), \tag{67}$$

where, as in equation (27) h is the number of successive plus twos immediately following the first element in $\lambda(\lambda')_{\lambda'}$; h_1 the number of successive plus twos immediately following α_{h+1}; h_2 the number immediately following α_{h+h_1+2}; ....; h_s the number immediately following $\alpha_{h+h_1+h_2+\dots+h_{s-k}+s-1}$; and where

$$\xi_t = \frac{1}{1-\lambda'} \cdots \frac{1}{\lambda'} + 1,\ t = 1,\ 2, \dots,\ h+h_1+h_2+\dots+h_{s-1}+s. \tag{68}$$

When

$$\xi_{h+h_1+\dots+h_{s-1}+s+1} = \left(\frac{1}{\lambda'}+1\right) \cdots \left(-\frac{1}{\lambda'}\right), \tag{69}$$

equation (67) may take the form

$$\lambda\left(\frac{\xi_{h+1}-1}{\xi_{h+1}-2}\right)_{\lambda'} = \Big(2_1, 2_2, \dots, 2_{\substack{a-3\\h+1}},\ 3+h_1,\ 2_1, \dots,\ 2_{\substack{1\\a-3\\h+h_1+2}},\ 3+h_2, \dots,\ 3+h_{s-1},\ 2_1,\ 2_2, \dots,\ 2_{\substack{a-2\\h+h_1+\dots+h_{s-1}+s}},\ -(\xi_{h+h_1+\dots+h_{s-1}+s+1}-1)\Big). \tag{70}$$

Returning now to equation (45) it may be written in the form

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = \Big(h+2,\ 2_1, \dots,\ 2_{\substack{a-3\\h+1}},\ 3+h_1,\ 2_1, \dots,\ 2_{\substack{a-3\\h+h_1+2}},\ 3+h_2, \dots,\ 3+h_{s-1},\ 2_1, \dots,\ 2_{\substack{a-3\\h+h_1+\dots+h_{s-1}+s}},\ 2+\frac{1}{\xi_{h+h_1+\dots+h_{s-1}+s+1}-1}\Big), \tag{71}$$

where condition (68) is satisfied. When in addition to this condition (70) is also satisfied, *i. e.*

$$\xi_{h+h_1+\dots+\underset{s-1}{h}+s+1} = \left(\frac{1}{\lambda'}+1\right)\dots\left(-\frac{1}{\lambda'}\right), \tag{72}$$

then equation (71) takes the form

$$\begin{aligned}\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = \Big(&h+2, 2_1, 2_2, \dots, \underset{h+1}{2_{\alpha-3}}, 3+h_1, 2_1, \dots, \underset{h+h_1+2}{2_{\alpha-3}},\\ &3+h_2, \dots, 3+h_{s-1}, 2_1, \dots, \underset{h+h_1+\dots+\underset{s-1}{h}+s}{2_{\alpha-2}}, -(\xi_{h+h_1+\dots+\underset{s-1}{h}+s+1}-1)\Big).\end{aligned} \tag{73}$$

In equations (69)–(72) $h, h_1, \dots, h_{s-1}$ preserve the meanings assigned under equation (67).

6.° Every case which may arise in the λ-development ($\lambda = \lambda'$) of $\frac{1}{\lambda'}$ has now been considered. The results are contained in equations (37), (44), (55), (60), (71), (73). From these equations may be drawn the following proposition.

THEOREM I. *$\lambda(\lambda')_{\lambda'}$ and $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$ from elements of certain ranks on differ in and only in the signs of their elements; or in $\lambda(\lambda')_{\lambda'}$ $\xi_t = \frac{1}{1-\lambda'}\cdots\frac{1}{\lambda'}+1$, $t = 1, 2, \dots n-1$, $0 < \lambda' \overset{=}{<} \tau$ aud $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$ is derived from $\lambda(\lambda')_{\lambda'}$ by the rule embodied in equation* (71).

7.° As a convenient notation not only for this section but also for the following let us write

$$\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'} = (\beta_1, \beta_2, \dots, \beta_{i-1}, \eta_i), \tag{74}$$

i a positive integer not zero.

Let η_t denote the complete quotient corresponding to β_t, $t = 0, 1, 2, i-1$. When η_t is an integer let equation (74) be uniformly written

$$\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'} = (\beta_1, \beta_2, \dots, \beta_{h},). \tag{75}$$

Since $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$, because of the symmetry of the interval system with respect to zero, is derived from $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$ simply by changing the signs of all the elements in the latter it follows at once by Theorem I that

Theorem II. $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$ *and* $\lambda(\lambda')_{\lambda'}$ *are ultimately alike or in* $\lambda(\lambda')_{\lambda'}$ $\xi_{t+1} < 0$, $|\xi_{t+1}| = \frac{1}{1-\lambda'} \cdots \frac{1}{\lambda'} + 1$, $t = 0, 1, \ldots, n-1$; $0 < \lambda' \leqq \tau$, *and the elements in* $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$ *differ from those in equation* (71) *only in sign.*

In theorems I. and II. note that n may be taken as great as we please, or, if not, that the element in $\lambda(\lambda')_{\lambda'}$ following α_k is ∞.

8.° a) $\lambda(1-\lambda')_{\lambda'}$.

There are two cases for consideration.

a$_1$) $1 - \lambda' < \lambda'$.

In this case $1 - \lambda' \sim 0$ and

$$\lambda(1-\lambda')_{\lambda'} = 0 - \cfrac{1}{\cfrac{1}{1-\lambda'}} = \left(0, -\frac{1}{1-\lambda'}\right) = (0, -\xi_1) \tag{76}$$

$$= (0, -\alpha_1, -\alpha_2, \ldots, -\alpha_{n-1}, -\xi_n).$$

a$_2$) $1 - \lambda' \geqq \lambda'$.

Here $1 - \lambda' \sim 1$ and

$$\lambda(1-\lambda')_{\lambda'} = \left(1, \frac{1}{\lambda'}\right) = (1, -\beta_1, -\beta_2, \ldots, -\beta_{n-1}, -\eta_n). \tag{77}$$

b) $\lambda\left(\frac{\lambda'}{\lambda'-1}\right)_{\lambda'}$.

In this case

$$\frac{\lambda'}{\lambda'-1} = -\left\{\frac{1}{1-\lambda'} - 1\right\} = -(\xi_1 - 1), \tag{78}$$

and since $\xi_1 > 1$ formula (23) applies. Hence

$$\lambda\left(\frac{\lambda'}{\lambda'-1}\right)_{\lambda'} = (-(\alpha_1 - 1), -\alpha_2, \ldots, -\alpha_{n-1}, -\xi_n). \tag{79}$$

c) $\lambda\left(\frac{\lambda'-1}{\lambda'}\right)_{\lambda'}$.

In this case

$$\frac{\lambda'-1}{\lambda'} = -\left(\frac{1}{\lambda'} - 1\right) \tag{80}$$

and, since $\frac{1}{\lambda'} \geqq 1$ formula (23) again applies. Hence

$$\lambda\left(\frac{\lambda'-1}{\lambda'}\right) = (-(\beta_1 - 1), -\beta_2, \ldots, -\beta_{n-1}, -\eta_n). \tag{81}$$

9.° It will be convenient here to complete the treatment of the case partially dealt with under 4°; viz., $\lambda(\xi_t - m)_{\lambda'}$, m any integer. Because of the symmetry of the interval system we need consider in detail only the case $m > \xi_t > 0$. There are three cases:

a) $\xi_t - \alpha_t = -1 \ldots -\lambda'$.

Now, since $\xi_t - \alpha_t \sim -1$

$$\xi_t - m \equiv \xi_t - \alpha_t - (m - \alpha_t) \sim -(m - \alpha_t + 1) \tag{82}$$

Hence

$$\lambda(\xi_t - m)_{\lambda'} = \left(-(m - \alpha_t + 1) - \cfrac{1}{\cfrac{1}{\xi_t - \alpha_t + 1}}\right) \tag{83}$$

$$= \left(-(m - \alpha_t + 1), -\frac{1}{\xi_t - \alpha_t + 1}\right).$$

Since

$$\xi_t - \alpha_t = -\frac{1}{\xi_{t+1}}$$

equation (83) becomes

$$\lambda(\xi_t - m)_{\lambda'} = \left(-(m - \alpha_t + 1), 1 + \frac{1}{\xi_{t+1} - 1}\right). \tag{84}$$

The last complete quotient comes under the case presented in equation (29), so that we may say that $\lambda(\xi_t - m)_{\lambda'}$ and $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$ are in the case under consideration ultimately alike.

b) $\xi_t - \alpha_t = (-\lambda') \ldots 0$.

In this case, since $\xi_t - \alpha_t \sim 0$, and $\xi_t - \alpha_t < 0$

$$\xi_t - m \equiv \xi_t - \alpha_t - (m - \alpha_t) \sim -(m - \alpha_t). \tag{85}$$

Hence

$$\lambda(\xi_t - m)_{\lambda'} = -(m - \alpha_t) - \cfrac{1}{\cfrac{1}{\xi_t - \alpha_t}} = \left(-(m - \alpha_t), -\frac{1}{\xi_t - \alpha_t}\right). \tag{86}$$

Since

$$\xi_t - \alpha_t = -\frac{1}{\xi_{t+1}}$$

equation (86) becomes

$$\lambda(\xi_t - m)_{\lambda'} = (-(m - \alpha_t), \alpha_{t+1}, \ldots, \alpha_{n-1}, \xi_n). \tag{87}$$

$$\text{c)}\quad \xi_t - \alpha_t = 0 \ldots . (\lambda').$$

This case is most conveniently treated under two sub-cases c$_1$) and c$_2$).

$$\text{c}_1)\quad \xi_t - \alpha_t - 1 \sim -1.$$

Now

$$\xi_t - m \equiv \xi_t - \alpha_t - (m - \alpha_t) \sim -(m - \alpha_t). \tag{88}$$

Hence

$$\begin{aligned}\lambda(\xi_t - m)_{\lambda'} &= \left(-(m - \alpha_t), -\frac{1}{\xi_t - \alpha_t}\right) = (-(m - \alpha_t), \xi_{t+1}) \\ &= (-(m - \alpha_t), \alpha_{t+1}, \alpha_{t+2}, \ldots, \alpha_{n-1}, \xi_n).\end{aligned} \tag{89}$$

$$\text{c}_2)\quad \xi_t - \alpha_t - 1 \sim 0.$$

In this case

$$\xi_t - m = \xi_t - \alpha_t - (m - \alpha_t) \sim -(m - \alpha_t - 1). \tag{90}$$

Therefore

$$\lambda(\xi_t - m)_{\lambda'} = \left(-(m - \alpha_t - 1), -\frac{1}{\xi_t - \alpha_t - 1}\right) = \left(-(m - \alpha_t - 1), \frac{\xi_{t+1}}{\xi_{t+1} + 1}\right). \tag{91}$$

It follows from the correspondence in equation (90) that

$$\left|\frac{\xi_{t+1}}{\xi_{t+1} + 1}\right| > 1. \tag{92}$$

Hence $\xi_{t+1} < 0$. Now writing equation (91) in the form

$$\lambda(\xi_t - m) = \left(-(m - \alpha_t - 1), 1 + \frac{1}{-\xi_{t+1} - 1}\right) \tag{93}$$

it is evident that its further reduction comes under the case considered in equation (29), where ξ_{h+1} is to be replaced by $-\xi_{t+1}$.

As already pointed out $\lambda(\xi_t - m)_{\lambda'}$, $\xi_t < 0, m > |\xi_t|$ can be derived from the cases just treated by considerations of symmetry. This, then completes the case partially treated under 4.° Accordingly we shall hereafter consider $\lambda(\xi_t + m)$, t any positive integer, m any integer, as known.

§3. λ-*developments* ($\lambda = \lambda'$, $0 < \lambda' \leqq 1$).

1.° The object of this section is to derive the λ-development ($\lambda = \lambda'$) of x_0, x_0 any real number not zero, to discuss certain properties of this development, and by means of these to set up a norm whereby to decide when a given terminating, or finite, development is a λ-development ($\lambda = \lambda'$). In a finite λ-development all the elements except possibly the last are integers. The last element may be integral, fractional or irrational, but except when otherwise specified, as in the last section following theorems I. and II., is in all cases finite.

2.° The finite λ-development ($\lambda = \lambda'$) of a real number x_0, ($x_0 \neq 0$), is derived from the equations arising when to i the values 0 to $n-1$ are successively assigned in the equation

$$x_i = a_i - \frac{1}{x_{i+1}}, \qquad x_i \sim a_i,\ a_i \text{ an integer}, \tag{1}$$

by the elimination of $x_1, x_2, \ldots, x_{n-1}$. The result in the usual notation is

$$\lambda\,(x_0)_{\lambda'} = (a_0, a_1, \ldots, a_{n-1}, x_n). \tag{2}$$

To examine more closely the values of $a_0, a_1, \ldots, a_{n-1}, x_n$ it is necessary to make explicit and precise the correspondence of x_i and a_i.

When the correspondence of x_i and a_i may be completely determined by the unmodified interval system (15), §1, always possible for $i = 0$, then

$$x_i - a_i = \operatorname{sgn.} x_i \{(-\lambda') \qquad \ldots\ldots (\lambda')\}, |x_i| < \lambda',\ 0 \leqq i < n; \tag{3}$$

$$x_i - a_i = \operatorname{sgn.} x_i \{ -\overline{1-\lambda'} \ldots\ldots (\lambda')\}, |x_i| \geqq \lambda',\ 0 \leqq i < n. \tag{4}$$

Since

$$x_i - a_i = -\frac{1}{x_{i+1}}$$

equations (3) and (4), respectively, give

$$x_{i+1} = \operatorname{sgn.} x_i \left\{\left(\frac{1}{\lambda'}\right) \ldots\ldots \left(-\frac{1}{\lambda'}\right)\right\}, |x_i| < \lambda',\ 0 \leqq i < n; \tag{5}$$

$$x_{i+1} = \operatorname{sgn.} x_i \left\{\frac{1}{1-\lambda'} \ldots\ldots \left(-\frac{1}{\lambda'}\right)\right\}, |x_i| \geqq \lambda',\ 0 \leqq i < n. \tag{6}$$

The intervals in equations (5) and (6) are composed of normal unit intervals and two special or end intervals. Each end interval has one limit number coincident with a limit of the interval under consideration in equation (5) or (6). The end intervals may in certain cases considered later be identical. In the discussion of equations (5) and (6) two cases present themselves according as normal unit intervals or end intervals are concerned. It will be convenient to consider first the case involving only normal intervals.

3.° *Normal intervals.*—When $|x_0| \geqq \lambda'$ and all complete quotients x_i, $i = 1, 2, \ldots, n$, lie in normal unit intervals, then equations (4) and (6) are satisfied for $i = 0, 1, 2, \ldots, n-1$. When $|x_0| < \lambda'$ and all complete quotients x_i, $i = 1, 2, \ldots, n-1$, lie in normal unit intervals then equations (3) and (5) are satisfied for $i = 0$, and equations (4) and (6) for $i = 1, 2, \ldots, n-1$. Since for $|x_i| < \lambda'$ $\operatorname{sgn.} x_{i+1} = -\operatorname{sgn.} x_i$, then every x_i, a_i, x_{i+1} satisfying equations (3) and (5) satisfy also equations (4) and (6) in the sense that x_i, a_i, x_{i+1} will be found in their respective intervals as assigned by equations (4) and (6). Adopting the convention that for $|x_i| < \lambda'$ sgn. $a_i =$ sgn. x_i, an equation necessarily true in all other cases, equations (4) and (6) may take the form

$$x_i - a_i = \text{sgn. } a_i \{ -\overline{1-\lambda'} \; \ldots \; (\lambda') \} \; i = 0, 1, 2, \ldots, n-1, \tag{7}$$

$$x_{i+1} = \text{sgn. } a_i \left\{ \frac{1}{1-\lambda'} \; \ldots \left(-\frac{1}{\lambda'} \right) \right\}, i = 0, 1, 2, \ldots, n-1, \tag{8}$$

where x_i and x_{i+1} lie in normal unit intervals.

4.° *The general case.*—To dispose of this case we begin by establishing the following equation:

$$x_{i+1} = \text{sgn. } a_f \{ Y_{i,r,f} \; \ldots \; Z_{i,s,f} \}, \; i = f, f+1, \ldots, n-1, \tag{9}$$

where r is the number of successive elements belonging to the left end interval and immediately preceding x_{i+1}, s the number of successive elements belonging to the right end interval and immediately preceding x_{i+1}, x_f corresponds to a_f and lies in a normal unit interval, and the left and the right end interval limits are to be interpreted by the following rules: for the left end interval limit

$$Y_{i,h,f} = \left\{ \begin{matrix} \xi_{h+1} \\ (-\eta_{h+1}) \end{matrix} \right\}^* \text{when sgn. } a_f \text{ sgn. } a_{i-h} = \left\{ \begin{matrix} +1 \\ -1 \end{matrix} \right\}; \tag{10}$$

for the right end interval limit

$$Z_{i,h,f} = \left\{ \begin{matrix} (\eta_{h+1}) \\ -\xi_{h+1} \end{matrix} \right\} \text{when sgn. } a_f \text{ sgn. } a_{i-h} = \left\{ \begin{matrix} +1 \\ -1 \end{matrix} \right\}, \; 0 \leqq h < i. \tag{11}$$

It should be remarked that r and s are simultaneously different from zero when and only when the left and the right end intervals are identical.

Let equation (9) be satisfied for $i = i_1$, then it is to be shown that it is satisfied when $i = i_1 + 1$. For $i = i_1$, equation (9) becomes

$$x_{i_1+1} = \text{sgn. } a_f \{ Y_{i,r_1,f} \; \ldots \; Z_{i_1 s_1,f} \}, \tag{12}$$

where $r = r_1$ $s = s_1$ for $i = i_1$.

a) Let a_{i_1+1} fall in the left end interval and not in the right.

Then, to work out the equations for sgn. $a_f = +1$,

$$x_{i_1+1} - a_{i_1+1} = \quad \xi_{r_1+1} - a_{i_1+1} \; \ldots \; (\lambda'), \quad a_{i_1+1} > 0; \tag{13}$$

$$x_{i_1+1} - a_{i_1+1} = (-\eta_{r_1+1} - a_{i_1+1}) \; \ldots \; (1-\lambda'), \; a_{i_1+1} < 0. \tag{14}$$

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[*] ξ_{h+1} and η_{h+1} have the meanings assigned in § 2.

The corresponding equations for x_{i_1+2} are, accordingly,

$$x_{i_1+2} = \quad \xi_{r_1+2} \dots \left(-\frac{1}{\lambda'}\right), \text{ since } \xi_{r_1+1} \sim a_{i_1+1},\ a_{i_1+1} > 0\,; \tag{15}$$

$$x_{i_1+2} = (-\eta_{r_1+2}) \dots \quad -\frac{1}{1-\lambda'}, \text{ since} -\eta_{r_1+1} \sim a_{i_1+1},\ a_{i_1+1} < 0\,. \tag{16}$$

In view of the symmetry of the interval system the complete result may be at once written

$$x_{i_1+2} = \text{sgn. } a_f \left\{ Y_{i_1+1, r_1+1, f} \dots Z_{i_1+1, 0, f} \right\}. \tag{17}$$

This is identical with equation (9) for $i = i_1 + 1$, $r = r_1 + 1$, $s = 0$.

b) Let x_{i_1+1} fall in the right end interval and not in the left. Then

$$x_{i_1+1} - a_{i_1+1} = \text{sgn. } a_f \left\{ -1 + \lambda' \dots Z_{i_1, s_1, f} - a_{i_1+1} \text{sgn. } a_f \right\},\ a_{i_1+1} > 0\,; \tag{18}$$

$$x_{i_1+1} - a_{i_1+1} = \text{sgn. } a_f \left\{ \quad (\lambda') \quad \dots Z_{i_1, s_1, f} - a_{i_1+1} \text{sgn. } a_f \right\},\ a_{i_1+1} < 0\,. \tag{19}$$

Since $Z_{i_1, s_1, f} \sim a_{i_1+1} \text{sgn. } a_f$ we have from equations (18) and (19)

$$x_{i_1+2} = \text{sgn. } a_f \left\{ Y_{i_1+1, 0, f} \dots Z_{i_1+1, s_1+1, f} \right\}, \tag{20}$$

an equation identical with equation (9) for $i = i_1 + 1$, $r = 0$, $s = s_1 + 1$.

c) x_{i_1+1} lies at the same time in both end intervals. Then

$$x_{i_1+1} - a_{i_1+1} = \text{sgn. } a_f \left\{ Y_{i_1, r_1, f} - a_{i_1+1} \text{sgn. } a_f \dots Z_{i_1, s_1, f} \dots - a_{i_1+1} \text{sgn. } a_f \right\} \tag{21}$$

and

$$x_{i_1+2} = \text{sgn. } a_f \left\{ Y_{i_1+1, r_1+1, f} \dots Z_{i_1+1, s_1+1, f} \right\}, \tag{22}$$

since

$$Y_{i_1, r_1, f} - a_{i_1+1} \text{ sgn. } a_f = -\frac{1}{Y_{i_1+1, r_1+1, f}} \tag{23}$$

and

$$Z_{i_1, s_1, f} - a_{i_1+1} \text{ sgn. } a_f = -\frac{1}{Z_{i_1+1, s_1+1, f}}. \tag{24}$$

Equation (22) is identical with equation (9) for $i = i_1 + 1$, $r = r_1 + 1$, $s = s_1 + 1$.

d) x_{i_1+1} falls in neither end interval but in a normal unit interval. Then the correspondence of x_{i_1+1} and a_{i_1+1} is completely determined by the interval system (15), §1. Hence (cf. equation (7))

$$x_{i_1+1} - a_{i_1+1} = \text{sgn. } a_f \left\{ -\overline{1-\lambda'} \dots (\lambda') \right\} \tag{25}$$

and

$$x_{i_1+2} = \text{sgn. } a_f \left\{ Y_{i_1+1, 0, f} \dots Z_{i_1+1, 0, f} \right\}, \tag{26}$$

The last equation is identical with equation (9) for $i = i_1 + 1$; $r = s = 0$.

Now it has been shown that if equation (9) holds for $i = i_1$ then it is true for $i = i_1 + 1$, $f \overset{=}{<} i \overset{=}{<} n - 1$. For $i = f$, equation (9) is identical with equation (8) and therefore a valid equation. Hence equation (9) is true also for $i = f + 1$, $f + 2, \dots n - 1$.

x_f lies in a normal interval certainly for $f=0$ and consequently equation (9) defines completely every complete quotient $x_1, x_2, \ldots, x_n$ in $\lambda\,(x_0)_{\lambda'}$ in terms of preceding representative integers and the complete quotients in $\lambda\,(\lambda')_{\lambda'}$ and in $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$.[*]

When in equation (13) $a_{i_1+1}=\alpha_k$, or, in equation (14) $a_{i_1+1}=-\beta_l$, then

$$x_{i_1+2}=\quad\infty\ \ldots.\left(-\frac{1}{\lambda'}\right), \tag{27}$$

$$x_{i_1+2}=(-\infty)\ \ldots.\quad -\frac{1}{1-\lambda'}. \tag{28}$$

Equations (27) and (28) may be derived directly from equations (15) and (16) by the convention $\xi_{k+1}=\eta_{l+1}=\infty$.[†] A like remark applies to equations (17)–(22), inclusive.

Let $Y_{i,h,f}$ correspond to $A_{i,h,f}$ and $Z_{i,h,f}$ to $B_{i,h,f}$. Then corresponding to equations (10) and (11) we have

$$A_{i,h,f}=\begin{Bmatrix}\alpha_{h+1}\\ -\beta_{h+1}\end{Bmatrix}\text{ when sgn. }a_f\text{ sgn. }a_{i-h}=\begin{Bmatrix}+1\\ -1\end{Bmatrix}; \tag{29}$$

and

$$B_{i,h,f}=\begin{Bmatrix}\beta_{h+1}\\ -\alpha_{h+1}\end{Bmatrix}\text{ when sgn. }a_f\text{ sgn. }a_{i-h}=\begin{Bmatrix}+1\\ -1\end{Bmatrix},\ 0\leqq h<i, \tag{30}$$

where r and s have the meanings assigned under equation (9) and x_f, corresponding to a_f, lies in a normal unit interval.

Hence, corresponding to equation (9) we may write

$$a_{i+1}=\text{sgn. }a_f\,\{A_{i,r,f}\ldots.B_{i,s,f}\}, \tag{31}$$

Equations (9) and (31) completely solve our first problem; viz., that of assigning to every real number x_0 a λ-development ($\lambda=\lambda'$). They are the fundamental equations of our theory, including equation (8) as a particular case.

The preceding results may be resumed in the following theorem:

Theorem I. *For x_0 a real non-zero number*

$$\lambda\,(x_0)_{\lambda'}=(a_0, a_1, \ldots, a_{n-1}, x_n),\ n>0,$$

---

[*] And herein, by way of remark, appears the fundamental importance of $\lambda\,(\lambda')_{\lambda'}$ and $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$ in every λ-development ($\lambda=\lambda'$).

[†] Cf. §1, 2°; also remark under theorem II, §2.

where

1.° $x_0 \sim a_0$,
2.° $a_{i+1} = sgn.\ a_0 \{A_{i,r,0} \ldots B_{i,s,0}\}$, $i = 0, 1, 2, \ldots, > n-2$,
3.° $x_n = sgn.\ a_0 \{Y_{i,r_1,0} \ldots Z_{i,s_1,0}\}$.

the expressions in the interval limits being interpreted by equations (10), (11), (29) *and* (30), *with the addition that when i takes the particular value n, r and s take respectively the particular values r_1 and s_1.*

5.° Employing the notation of theorem I we state the converse proposition:

THEOREM II. *Conversely, when for a continued fraction*

$x_0 = (a_0, a_1, \ldots, a_{n-1}, x_n)$, $n > 0$, a_i, $i = 0, 1, 2, \ldots, n-1$, *an integer, the following conditions are satisfied:*

1.° $a_{i+1} = sgn.\ a_0 \{A_{i,r,0} \ldots B_{i,s,0}\}$, $i = 0, 1, 2, \ldots, n-2$.
2.° $x_n = sgn.\ a_0 \{Y_{i,r_1,0} \ldots Z_{i,s_1,0}\}$

then

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \ldots, a_{n-1}, x_n).$$

The proof is as follows:

a) $n = 1$.

Condition 2° asserts the correspondence of $a_0 - \frac{1}{x_1}$ and a_0. Therefore, since $x_0 = a_0 - \frac{1}{x_1}$.

$$\lambda(x_0)_{\lambda'} = (a_0, x_1). \tag{32}$$

b) $n > 1$.

The interval assigned to x_{n-1} in accordance with theorem II. is

$$x_{n-1} = \text{sgn.}\ a_0 \{Y_{n-2,r_2,0} \ldots Z_{n-2,s_2,0}\}, \tag{33}$$

where r_2 and s_2 are the values of r and s for $i = n-1$.

By condition 1° for $i = n-2$,

$$a_{n-1} = \text{sgn.}\ a_0 \{A_{n-2,r_2,0} \ldots B_{n-2,s_2,0}\} \tag{34}$$

and by condition 2° for $i = n$

$$-\frac{1}{x_n} = \text{sgn.}\ a_0 \left\{-\frac{1}{Y_{n-1,r_1,0}} \ldots -\frac{1}{Z_{n-1,s_1,0}}\right\}. \tag{35}$$

b₁) $r_1 \neq 0$, $s_1 \neq 0$.

Therefore, $r_1 = r_2 + 1$, $s_1 = s_2 + 1$.

Hence

$$a_{n-1} - \frac{1}{x_n} = \text{sgn.}\ a_0 \left\{A_{n-2,r_2,0} - \frac{1}{Y_{n-1,r_1,0}} \ldots B_{n-2,s_2,0} - \frac{1}{Z_{n-1,s_1,0}}\right\} \tag{36}$$
$$= \text{sgn.}\ a_0 \{Y_{n-2,r_2,0} \ldots Z_{n-2,s_2,0}\},$$

this last reduction being made by means of the correspondence of $Y_{n-2,r_2,0}$ and $A_{n-2,r_2 0}$ on the one hand and of $Z_{n-2,s_2,0}$ and $B_{n-2,s_2,0}$ on the other. Since $x_{n-1} = a_{n-1} - \frac{1}{x_n}$, x_{n-1} lies in the interval required by equation (33).

$$\text{b}_2) \quad r_1 > 0,\ s_1 = 0.$$

In this case $r_1 = r_{2+1}$ and a_{n-1} lies in the left end interval, equation (34), and not in the right. Under this hypothesis

$$-\frac{1}{x_n} = \text{sgn.}\ a_0 \left\{ -\frac{1}{Y_{n-1,r_1,0}} \cdots -\frac{1}{Z_{n-1,0,0}} \right\}, \tag{37}$$

and, consequently

$$a_{n-1} - \frac{1}{x_n} = \text{sgn.}\ a_0 \left\{ A_{n-2,r_2,0} - \frac{1}{Y_{n-1,r_1,0}} \cdots A_{n-2,r_2,0} - \frac{1}{Z_{n_1 0,0}} \right\}, \tag{38}$$

where $A_{n-2,r_2,0}$ is to be interpreted by equation (29).

The left interval limit in equation (38) reduces at once to the left interval limit in equation (33). We proceed to show by evaluating the right interval limit that the interval in equation (38) always lies in or at most coincides with the interval in equation (33). Since a_{n-1} lies in the left end interval of equation (33) and not in the right, the right limit number must certainly lie without the normal unit interval associated with a_{n-1}. We now show that the interval in equation (38) contains at most one normal unit interval.

For the right interval limit we have

$$\begin{aligned}
&(\alpha_{r_2+1} + \lambda'), && \text{when } a_{n-1} = \alpha_{r_2+1}\,\text{sgn.}\ a_0 \text{ and sgn.}\ a_0 = \text{sgn.}\ a_{n-2-r} = \text{sgn.}\ a_{n-1};\\
&\alpha_{r_2+1} + 1 - \lambda', && \text{when } a_{n-1} = \alpha_{r_2+1}\,\text{sgn.}\ a_0 \text{ and sgn.}\ a_0 = \text{sgn.}\ a_{n-2-r_2} = -\text{sgn.}\ a_{n-1};\\
&(\beta_{r_2+1} + \lambda'), && \text{when } a_{n-1} = \beta_{r_2+1}\,\text{sgn.}\ a_0 \text{ and sgn.}\ a_0 = -\text{sgn.}\ a_{n-2-r_2} = \text{sgn.}\ a_{n-1};\\
&\beta_{r_2+1} + 1 - \lambda', && \text{when } a_{n-1} = \beta_{r_2+1}\,\text{sgn.}\ a_0 \text{ and sgn.}\ a_0 = -\text{sgn.}\ a_{n-2-r_2}\\
& && \qquad = -\text{sgn.}\ a_{n-1};
\end{aligned}$$

In every instance the right interval limit in equation (33) coincides with one limit of the normal unit interval represented by a_{n-1}, while the left interval limit, as already pointed out, is identical with the left interval limit of equation (33), which always lies in the normal unit interval associated with a_{n-1}, or coincides with a limit number of this interval. Hence the interval in equation (38) is at most coextensive with a normal unit interval and is included in the interval in equation (33). Hence in this case $a_{n-1} - \frac{1}{x_n}$ $(\equiv x_{n-1})$ is in the required interval.

b$_3$) $r_1 = 0,\ s_1 > 0$.

Here $s_1 = s_2 + 1$ and a_{n-1} lies in the right end interval and not in the left. This case differs so slightly from the preceding that it is presented in merest outline. The interval equation for $a_{n-1} - \frac{1}{x_n}$ corresponding to equation (38) will have its right limit number identical with the right limit number in equation (33). The left limit numbers save as to order are identical with the right limit numbers found in the preceding case. Hence the conclusion in this case as in the preceding that $a_{n-1} - \frac{1}{x_n}$ $(\equiv x_{n-1})$ lies in the interval of equation (33).

b$_4$) $r_1 = s_1 = 0$.

In this case a_{n-1} lies in a normal unit interval, *i. e.*, the interval assigned to x_{n-1} by equation (33) contains in this case certainly one normal unit interval; viz., that associated with a_{n-1}. It is to be shown that $a_{n-1} - \frac{1}{x_n}$ lies in this normal unit interval.

For this case

$$a_{n-1} - \frac{1}{x_n} = \text{sgn. } a_0 \left\{ a_{n-1} \text{ sgn. } a_0 - \frac{1}{Y_{n-1,0,0}} \cdots a_{n-1} \text{ sgn. } a_0 - \frac{1}{Z_{n-1,0,0}} \right\}. \quad (39)$$

When sgn. a_0 sgn. $a_{n-1} = +1$, equation (39) becomes

$$a_{n-1} - \frac{1}{x_n} = \text{sgn. } a_0 \{ \overline{a_{n-1} - 1 + \lambda'} \text{ sgn. } a_0 \cdots (a_{n-1} + \lambda') \text{ sgn. } a_0 \}; \quad (40)$$

when sgn. a_0 sgn. $a_{n-1} = -1$, equation (39) becomes

$$a_{n-1} - \frac{1}{x_n} = \text{sgn. } a_0 \{ (a_{n-1} - \lambda') \text{ sgn. } a_0 \cdots \overline{a_{n-1} + 1 - \lambda'} \text{ sgn. } a_0 \}. \quad (41)$$

Hence by equations (40) and (41) $a_{n-1} - \frac{1}{x_n}$ $(\equiv x_{n-1})$ lies in the normal unit interval associated with a_{n-1}, and hence, also, in the interval of equation (33).

Having considered the several cases of our theorem, which may arise under b), we may now conclude that

$$x_0 = (a_0, a_1, \ldots, a_{n-2}, x_{n-1}). \quad x_{n-1} = a_{n-1} - \frac{1}{x_n},$$

satisfies the conditions

1.° $a_{i+1} = \text{sgn. } a_0 \{ A_{i,r,0} \cdots B_{i,s,0} \}$, $i = 0, 1, 2, \ldots, n-3$,
2.° $x_{n-1} = \text{sgn. } a_0 \{ Y_{i,r_2,0} \cdots Z_{i,s_2,0} \}$.

This reduction may be repeatedly applied until we have finally

$$x_0 = (a_0, x_1),$$

a case considered under a). The result of this investigation is that x_i and a_i, $i = 0, 1, 2, \ldots, n-1$, correspond, when x_i is calculated by the equation

$$x_i = a_i - \frac{1}{x_{i+1}}.$$

This correspondence proves that the continued fraction in question is the λ-development $(\lambda = \lambda')$ of x_0 and, therefore, establishes theorem II.

This completes the second problem proposed; namely, to establish a criterion whereby to decide if a given finite continued fraction be a λ-development $(\lambda = \lambda')$.

6.° A few properties of λ-developments $(\lambda = \lambda')$ implicit in the preceding formulas are here made explicit for the purposes of the further discussion of our theory.

Since the end intervals are at most coextensive with normal unit intervals then when $x_i \sim a_i$ and lies on the zero side of a_i

$$|x_i - a_i| \leqq 1 - \lambda'; \tag{42}$$

and when x_i lies on the infinity side of a_i

$$|x_i - a_i| < \lambda' \qquad i = 0, 1, \ldots, n-1. \tag{43}$$

Hence, for the corresponding cases,

$$|x_{i+1}| \geqq \frac{1}{1-\lambda'} = \xi_1, \tag{44}$$

$$|x_{i+1}| > \frac{1}{\lambda'} = |\eta_0|. \tag{45}$$

Now when x_i lies on the $\left\{\begin{matrix}\text{zero}\\ \text{infinity}\end{matrix}\right\}$ side of a_i, sgn. $x_{i+1} = \left\{\begin{matrix}+\\ -\end{matrix}\right\}$ sgn. x_i, and, conversely. Evidently, also, when x_i lies on the $\left\{\begin{matrix}\text{zero}\\ \text{infinity}\end{matrix}\right\}$ side of a_i, sgn. a_{i+1} $= \left\{\begin{matrix}+\\ -\end{matrix}\right\}$ sgn. a_i, and conversely. Further (cf. §2, equation (3))

$$\xi_1 > 1 + \lambda' \tag{46}$$

and, hence

$$a_1 \geqq 2.$$

Therefore,

Theorem III. *When sgn.* $a_{i+1} =$ *sgn.* a_i, $|a_{i+1}| \geqq 2$, $i = 0, 1, 2, \ldots, n-1$.

Since

$$|\eta_0| = \frac{1}{\lambda'} \geqq 1,$$

then

Theorem IV. *When sgn.* $a_{i+1} = -$ *sgn.* a_i, $|a_{i+1}| \geqq 1$, $i = 0, 1, 2, \ldots, n-1$.

And, consequently

THEOREM V. *When* $|a_{i+1}| = 1$, *sgn.* $a_{i+1} = -$ *sgn.* a_i, $i = 0, 1, \ldots, n-1$.

When

$$\lambda' \leqq -\frac{1+\sqrt{5}}{2} \equiv \tau,$$

then

$$\frac{1}{\lambda'} \geqq 1 + \lambda';$$

and, when

$$\lambda' > \tau,$$

then

$$\frac{1}{\lambda'} < 1 + \lambda'.$$

In consequence theorem V may be stated thus:

THEOREM VI. *When sgn.* $a_{i+1} = -$ *sgn.* a_i *then for* $\lambda' > \tau_1$, $|a_{i+1},| \geqq 1$ *and for* $\lambda' \leqq \tau_1$, $|a_{i+1}| \geqq 2$.

When in equation (17), §2,

$$a_1 = a_2 = \ldots = a_j = 2,$$

then

$$\xi_{j+1} = \frac{1 - j\lambda'}{1 - (j+1)\lambda'}.$$

Now for

$$a_{f+1} = a_1, \quad a_{f+2} = a_2, \ldots, \quad a_{f+j} = a_j$$

equation (9) becomes

$$x_{i+1} = \text{sgn. } a_f \{Y_{i,j,f} \ldots Z_{i,j,f}\}, \; i = f + j, \tag{47}$$

or

$$x_{i+1} = \text{sgn. } a_f \left\{\frac{1 - j\lambda'}{1 - (j+1)\lambda'} \cdots \left(-\frac{1}{\lambda'}\right)\right\}. \tag{48}$$

When $j < \frac{1}{\lambda'} \leqq j + 1$, sgn. $x_{i+1} = -$ sgn. a_i. Hence

THEOREM VII. *The maximum number of syn.* a_f . *2's immediately preceded by* a_f, *the representative integer of a normal unit interval, and followed by an element with the same sign as* a_f, *which may enter a* λ-*development* ($\lambda = \lambda'$) *is* $j - 1$ *where*

$$j < \frac{1}{\lambda'} \leqq j + 1.$$

When the convention is adopted that sgn. $\infty =$ *sgn.* a_f, *then the maximum number is* j.

When

$$\frac{1}{\lambda'} = j + 1,$$

then

$$\lambda(\lambda')_{\lambda'} = (1, 2_1, 2_2, \ldots, 2_j).$$

In this case we may say

THEOREM VIII. *When in* $\lambda\,(x_0)_{\lambda'}$, $\frac{1}{\lambda'} = j + 1$ *there enters a maximum number of successive 2's having the sign of* a_f *then they are immediately followed by an element with the opposite sign or by* ∞.

For those values of λ' which permit $+\,2$ to enter a λ-development $(\lambda = \lambda')$ preceded by a positive element the sequence

$$(a_0,\ 2_1,\ 2_2,\ \ldots,\ 2_{j-2};\ s,\ 2_1,\ 2_2,\ \ldots,\ 2_{j-2},\ s,\ 2_1,\ 2_2,\ \ldots\ 2_{j-2},\ s,\ x_n) \qquad (49)$$

is a λ-development $(\lambda = \lambda')$, where $a_0 > 0$, $s = a_2 + 1$, a_2 the minimum positive integer which may follow $j - 2$ successive plus twos without requiring an immediate change of sign, j is defined by the inequality

$$j < \frac{1}{\lambda'} \leqq j + 1,$$

and

$$x_n = \xi_1 \ldots (\eta_1).$$

The truth of this statement is evident from theorem II.

§4. *Convergents of λ-developments* (λ = λ′).

1.° When p_{i-1}, p_i, p_{i+1}, and q_{i-1}, q_i, q_{i+1} satisfy the equations

$$p_{i+1} = a_{i+1}\,p_i - p_{i-1}, \quad p_{-1} = 1, \quad p_0 = a_0, \qquad (1)$$

$$q_{i+1} = a_{i+1}\,q_i - q_{i-1}, \quad q_{-1} = 0, \quad q_0 = 1, \quad i = 0, 1, 2, \ldots, n-2. \qquad (2)$$

where a_{i+1} is an element in the continued fraction

$$\lambda\,(x_0)_{\lambda'} = (a_0,\ a_1,\ \ldots,\ a_{n-1},\ x_n), \qquad (3)$$

then it may be shown that

$$\frac{p_i}{q_i},\ i = 0, 1, 2, \ldots, n-1, \qquad (4)$$

is the $(r+1)^{\text{th}}$ convergent of the continued fraction (3).

Directly by equation (1) follows

LEMMA I. *When* $|\,p_i\,| > |\,p_{i-1}\,|$, *then sgn.* $p_{i+1} =$ *sgn.* a_{i+1} . *sgn.* p_i, $i = 0, 1, \ldots, n-1$.

And, again,

LEMMA II. *When* $|p_{i+1}| \geqq |p_i| \geqq |p_{i-1}|$, *then* $|p_{i+2}| \geqq |p_{i+1}|$, $i = 0, 1, \ldots, n-3$.

For, when $|a_{i+2}| \geqq 2$, the conclusion of lemma II follows at once from the hypothesis by equation (1). When $|\,a_{i+2}\,| = 1$, then sgn. $a_{i+2} = -$ sgn. a_{i+1} and, lemma I,

$$|p_{i+2}| = |p_{i+1}| + |p_i| > p_{i+1}.$$

Now

$$p_0 = a_0,\ p_1 = a_1 p_1 - 1,\ p_2 = a_2 p_1 - p_0,\ \ldots, \tag{5}$$

and in all cases

$$|p_2| \geqq |p_1| \geqq |p_0|. \tag{6}$$

Hence, by virtue of lemma II,

Lemma III.

$$|p_{i+1}| \geqq |p_i|, \qquad i = 0, 1, 2, \ldots, n-2.$$

The inequality sign prevails in all cases in the lemma except as indicated below. When $\lambda(x_0)_{\lambda'}$ or $\lambda(-x_0)_{\lambda'}$ takes the form

$$(0, a_1, 1, \ldots) \quad a_1 < 0 \tag{7}$$

the equality sign is taken for $i = 0, 1, 2, \ldots h$, or when $\lambda(x_0)_{\lambda'}$ or $\lambda(-x_0)_{\lambda'}$ takes the form

$$(1, 2_1, \ldots), \tag{8}$$

the equality sign is taken for $i = 0, 1, 2, \ldots h$, where h is the number of successive plus "twos" following immediately the element one in the continued fractions one (7) and (8). h does not exceed $\left\{\begin{matrix} j-1 \\ j \end{matrix}\right\}$ where

$$\left\{\begin{matrix} j < \dfrac{1}{\lambda'} < j+1 \\ j+1 = \dfrac{1}{\lambda'} \end{matrix}\right\}^{*}. \tag{9}$$

By precisely similar reasoning on equation (2) we have corresponding to lemmas I., II., and III.

Lemma IV. *When* $|q_{i+1}| > |q_i|$, *then* $sgn.\ q_{i+1} = sgn.\ a_{i+1}\, sgn.\ q_i$, $i = 0, 1, 2, \ldots, n-2$.

Lemma V. *When* $|q_{i+1}| \geqq |q_i| \geqq |q_{i-1}|$, *then* $|q_{i+2}| > |q_{i+1}|$, $i = 1, 2, \ldots, n-3$.

Lemma VI. $|q_{i+1}| \geqq |q_i|$, $i = 0, 1, 2, \ldots, n-2$.

In lemma VI the equality sign is to be taken for $i = 0, 1$ when $\lambda(x_0)_{\lambda'}$ or $\lambda(-x_0)_{\lambda'}$ assumes the form

$$(0, -1, 1_1 \ldots). \tag{10}$$

2.° As in the theory of simple continued fractions it can be shown that

$$x_0 = \frac{p_i x_{i+1} - p_{i-1}}{q_i x_{i+1} - q_{i-1}}, \qquad 0 \leqq i \leqq n-1, \tag{11}$$

where x_0 is given by equation (3). By equation (11) after a slight reduction

$$x_0 - \frac{p_i}{q_i} = \frac{-1}{q_i^2\left(x_{i+1} - \dfrac{q_{i-1}}{q_i}\right)}. \tag{12}$$

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* Cf. Theorem VII, §3.

When $\lambda(x_0)_{\lambda'}$ terminates with a complete quotient $x_n = a_n$, x_{n+1} is to be taken as infinite. In this instance, since $\frac{q_{i-1}}{q_i}$ is finite, equation (12) gives

Hence
$$x_0 - \frac{p_n}{q_n} = 0. \tag{13}$$

Theorem I. *The value of a terminating λ-development ($\lambda = \lambda'$) may be expressed by a rational fraction.*

3.° When $\lambda(x_0)_{\lambda'}$ does not terminate with $x_n = a_n$ however great n may be, there are two cases to

a) $\lambda' = (0) \ldots . (1)$.

When sgn. $a_{i+1} =$ sgn. a_1, $|x_{i+1}| \geqq \frac{1}{1-\lambda'} > 1 + \lambda'$. And, since

then by equation (12)
$$\frac{q_{i-1}}{q_i} \leqq 1,$$

$$\left| x_0 - \frac{p_i}{q_i} \right| < \frac{1}{q_i^2 \lambda'}, \quad i = 0, 1, 2, \ldots, n-1. \tag{14}$$

When sgn. $a_{i+1} = -$ sgn. a_i, $|x_{i+1}| > \frac{1}{\lambda'}$. Let $\lambda' = 1 - \varepsilon$. Then

$$\frac{1}{\lambda'} > 1 + \varepsilon.$$

Hence in this case by equation (12)

$$\left| x_0 - \frac{p_i}{q_i} \right| < \frac{1}{q_i^2 \varepsilon}, \quad 0 \leqq i \leqq n-1. \tag{15}$$

b) $\lambda' = 1$.

In this case sgn. $x_{i+1} = -$ sgn. $a_i \equiv -$ sgn. $\frac{q_{i-1}}{q_i}$ (lemma IV.) and consequently

$$\left| x_{i+1} - \frac{q_{i-1}}{q_i} \right| = |x_{i+1}| + \left| \frac{q_{i-1}}{q_i} \right| > 1. \quad 0 \leqq i \leqq n-1. \tag{16}$$

Hence
$$\left| x_0 - \frac{p_i}{q_i} \right| < \frac{1}{q_i^2}, \quad 0 \leqq i \leqq n-1. \tag{17}$$

Now since $|q_i|$ increases without limit with i, an immediate inference from lemma IV. and following remark, i may be chosen so great that the second members of the inequalities (14), (15) and (17) become less than any given non-zero positive number however small. Therefore in all cases

$$\underset{i=\infty}{L} \frac{p_i}{q_i} = x_0. \tag{18}$$

4.° Consider now the non-terminating continued fraction

$$(a_0, a_1, \dots, a_i, a_{i+1}, \dots, a_{i+n}, \dots), \tag{19}$$

whose first $i + n + 1$ elements however great i and n may be satisfy for $\lambda = \lambda'$ the condition

$$a_{j+1} = \text{sgn.}\, a_0\{A_{j,r,0} \dots B_{j,s,0}\}, j = 0, 1, \dots, i+n, \tag{20}$$

where r and s are defined under equation (9) and $A_{j,r,0}$, $B_{j,s,0}$ under equations (29) and (30) of §3. Consider, also, the associated sequence of successive convergents derived from the p's and q's arising out of equation (19); viz.,

$$\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_i}{q_i}, \frac{p_{i+1}}{q_{i+1}}, \dots, \frac{p_{i+n}}{q_{i+n}}, \dots \tag{21}$$

This sequence, as will be shown, always has a limit. This limit will be taken by definition for the value of the non-terminating continued fraction (19). The λ-development ($\lambda = \lambda'$) of the value so defined of this infinite λ-development ($\lambda = \lambda'$) is, with exceptions to be specified, identically so far as partial quotients are concerned, the continued fraction (19).

It can be shown that

$$\frac{p_{i+n}}{q_{i+n}} - \frac{p_i}{q_i} = -\left\{\frac{1}{q_i q_{i+1}} + \frac{1}{q_{i+1} q_{i+2}} + \dots + \frac{1}{q_{i+n-2} q_{i+n-1}} + \frac{1}{q_{i+n-1} q_{i+n}}\right\}; \tag{22}$$

and, consequently, that

$$\left|\frac{p_{i+n}}{q_{i+n}} - \frac{p_i}{q_i}\right| \overset{=}{<} \frac{1}{|q_i| \cdot |q_{i+1}|} + \frac{1}{|q_{i+1}| \cdot |q_{i+2}|} + \dots + \frac{1}{|q_{i+n-2}| \cdot |q_{i+n-1}|} + \frac{1}{|q_{i+n-1}| \cdot |q_{i+n}|}. \tag{23}$$

Hence, since for $i > 2$ the q's increase in absolute value, for $i > 2$

$$\left|\frac{p_{i+n}}{q_{i+n}} - \frac{p_i}{q_i}\right| \overset{=}{<} \frac{1}{N(N+1)} + \frac{1}{(N+1)(N+2)} + \dots + \frac{1}{(N+n-2)(N+n-1)} + \frac{1}{(N+n-1)(N+n)} \overset{=}{<} \frac{1}{N} - \frac{1}{N+n}, \tag{24}$$

where $N = |q_i|$. Now i may be chosen so great that for all values of n,

$$\frac{1}{N} - \frac{1}{N+n} < \varepsilon$$

where ε is an arbitrarily assigned non-zero positive number. In truth when i is so chosen that $\frac{1}{N} < \varepsilon$, then for all values of n

$$\left| \frac{p_{i+n}}{q_{i+n}} - \frac{p_i}{q_i} \right| < \varepsilon .$$

Hence the sequence (21) has a limit. Let us call this limit x_0. Then by definition

$$x_0 = (a_0, a_1, \ldots, a_i, a_{i+1}, \ldots, a_{i+n}, \ldots). \tag{25}$$

5.° It will now be shown, with exceptions to be noted that

$$\lambda (x_0)_{\lambda'} = (a_0, a_1, \ldots, a_i, a_{i+1}, \ldots, a_{i+n}, x_{i+n+1}) \tag{26}$$

for all values of i and n however great.

By analogy to the definition in equation (25) x_j is defined by the equation

$$x_j = (a_j, a_{j+1}, \ldots, a_{i+n}, \ldots), \quad j = 0, 1, 2, \ldots, i+n, \ldots). \tag{27}$$

Let $\frac{P_i}{Q_i}$ be defined by the equation

$$\frac{P_i}{Q_i} = (a_1, a_2, \ldots, a_i). \tag{28}$$

Now when a_i satisfies the relation

$$a_i = \text{sgn.}\, a_0 \{ A_{i-1, r, 0} \ldots B_{i-1, s, 0} \}, \quad \text{(theorem II, §3)}, \tag{29}$$

then

$$(a_0, a_1, \ldots a_i) \tag{30}$$

is the λ-development $(\lambda = \lambda')$ of $\frac{p_i}{q_i}$. Hence

$$\frac{p_i}{q_i} = a_0 - \frac{1}{\frac{P_i}{Q_i}}, \tag{31}$$

where

$$\frac{p_i}{q_i} \sim a_0 .$$

a) Let it now be granted that there is no integer n_1 such that for all values of i, $i > n_1$, sequence (30) is not the $\lambda \left(\frac{p_i}{q_i} \right)_{\lambda'}$. Then taking limits of both members of equation (31) for $i = \infty$, we have

$$x_0 = a_0 - \frac{1}{x_1}, \tag{32}$$

where, because of the correspondence of $\frac{p_i}{q_i}$ and a_0 for certain values of i under the conditions imposed,

$$x_0 - a_0 = \text{sgn.}\, a_0 \{ -1 + \lambda' \ldots \lambda' \}. \tag{33}$$

hen
$$x_0 = a_0 + \lambda' \text{ sgn. } a_0, \tag{34}$$
does not correspond to a_0; in all other cases $x_0 \sim a_0$.

When $x_0 \sim a_0$, then it may be shown that either $x_1 \sim a_1$ or $x_1 = a_1 + \lambda' \text{ sgn. } a_1$.
d, again, when $x_1 \sim a_1$ then $x_2 \sim a_2$, or $x_2 = a_2 + \lambda' \text{ sgn. } a_2$, etc.

Hence equation (26) is true or else for some value of j
$$x_j = a_j + \lambda' \text{sgn. } a_j \qquad 0 \leqq j \leqq i + n. \tag{35}$$

b) Let it now be supposed that there is an integer n_1 such that for all lues of i, $i > n$, sequence (30) is not $\lambda\left(\frac{p_i}{q_i}\right)$. In this case select a number such that the sequence
$$(a_0, a_1, \ldots, a_i + \nu_i) \tag{36}$$
a λ-sequence ($\lambda = \lambda'$). ν_i may at the same time satisfy the inequalities
$$0 < |\nu_i| \leqq 1.$$

Now let $\frac{p'_i}{q'_i}$ be the $(i + 1)^{\text{th}}$ convergent of the sequence (36). Then
$$\frac{p'_i}{q'_i} - \frac{p_i}{q_i} = \frac{p_{i-1}(a_i + \nu_i) - p_{i-2}}{q_{i-1}(a_i + \nu_i) - q_{i-1}} - \frac{p_i}{q_i} = \frac{p_i + p_{i-1}\nu_i}{q_i + q_{i-1}\nu_i} - \frac{p_i}{q_i}. \tag{37}$$
Hence
$$\frac{p'_i}{q'_i} - \frac{p_i}{q_i} = \frac{\nu_i}{q_i(q_i + q_{i+1}\nu_i)}. \tag{38}$$

Passing to limits for $i = \infty$, we have
$$\underset{i=\infty}{L} \frac{p'_i}{q'_i} = \underset{i=\infty}{L} \frac{p_i}{q_i} = x_0. \tag{39}$$

Now by theorem II, §3, $\frac{p'_i}{q'_i} \sim a_0$. Hence
$$x_0 - a_0 = \text{sgn. } a_0 \{-1 + \lambda' \ldots . \lambda'\} \tag{40}$$
d either $x_0 \sim a_0$, or
$$x_0 = a_0 + \lambda' \text{ sgn. } a_0. \tag{41}$$

When $x_0 \sim a_0$, then it may be shown as in the case of x_0 either that $\sim a_1$ or
$$x_1 = a_1 + \lambda' \text{ sgn. } a_1.$$

When $x_1 \sim a_1$, then, again, it may be shown that either $x_2 \sim a_2$, or
$$x_2 = a_2 + \lambda' \text{ sgn. } a_2.$$

By continuing this reasoning the conclusion is reached that equation (26) is true or that for some value of j

$$x_j = a_j + \lambda' \,\mathrm{sgn.}\, a_j, \quad 0 \leqq j \leqq i + n. \tag{42}$$

Equations (35) and (42) are formally identical. Evidently

$$\lambda(x_j)_{\lambda'} = (a_j + \mathrm{sgn.}\, a_j,\ \alpha_1 \mathrm{sgn.}\, a_j,\ \alpha_2 \mathrm{sgn.}\, a_j,\ \alpha_3 \mathrm{sgn.}\, a_j,\ \ldots.). \tag{43}$$

Therefore, when equation (26) is not true

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \ldots., a_{j-1}, a_j + \mathrm{sgn.}\, a_j,\ \alpha_1 \mathrm{sgn.}\, a_j,\ \alpha_2 \mathrm{sgn}\, a_j,\ \alpha_3 \mathrm{sgn.}\, a_j,\ \ldots.),\quad 0 \leqq j \leqq i + n. \tag{44}$$

It has now been shown that $\lambda(x_0)_{\lambda'}$ is identical with sequence (19) for as many successive partial quotients beginning with the first, as we please or that it takes the particular form (44).

As an example of an exception consider the infinite continued fraction

$$(2, -3, -3_2, -3_3, \ldots.), \tag{45}$$

in which -3_j, $j = 0, 1, 2, \ldots., n$, n finite, satisfies equation (20) for

$$\lambda' = \tau \equiv \frac{-1 + \sqrt{5}}{2}.$$

For $n = \infty$, however, by definition we have from sequence (45)

$$x_0 = 2 + \tau, \tag{46}$$

whence

$$\lambda(x_0)_\tau = (3, 3, 3, 3, 3_3 \ldots.). \tag{47}$$

5.° A method of approximate calculation should provide a ready means of finding a sequence of numbers approaching a given number, of determining the error made by taking one of these numbers for the given number, and, finally, of making the approximation as close as may be desired.

When in the decimal approximation to a given number, a certain number of significant figures are taken, then a number of significant figures greater by one, and then, greater by two, etc., in the sequence of numbers thus formed each is a closer approximation than the preceding and the error committed by taking any term of this sequence for the given number is less than one unit of the order of the last significant figure to the right in the term chosen. Finally this error may be made as small as may be desired simply by making the number of significant figures in the approximate value sufficiently great.

To derive the corresponding properties of λ-developments ($\lambda = \lambda'$) we start from the equation

$$x_0 - \frac{p_i}{q_i} = \frac{-1}{q_i^2\left(x_{i+1} - \frac{q_{i-1}}{q_i}\right)}. \tag{48}$$

But

$$x_0 - \frac{p_{i+1}}{q_{i+1}} = x_0 - \frac{p_i}{q} - \left(\frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i}\right) = x_0 - \frac{p_i}{q_i} + \frac{1}{q_i\, q_{i+1}}. \tag{49}$$

Now by equation (48), since also

$$q_{i+1} = a_{i+1}\, q_i - q_{i-1}, \tag{50}$$

we have

$$\begin{aligned} x_0 - \frac{p_{i+1}}{q_{i+1}} &= \frac{-1}{q_i^2\left(x_{i+1} - \frac{q_{i-1}}{q_i}\right)} + \frac{1}{q_i^2\left(a_{i+1} - \frac{q_{i-1}}{q_i}\right)} \\ &= \frac{x_{i+1} - a_{i+1}}{q_i^2\left(x_{i+1} - \frac{q_{i-1}}{q_i}\right)\left(a_{i+1} - \frac{q_{i-1}}{q_i}\right)}. \end{aligned} \tag{51}$$

Since

$$x_{i+1} - a_{i+1} = -\frac{1}{x_{i+2}},$$

the last equation may take the form

$$x_0 - \frac{p_{i+1}}{q_{i+1}} = \frac{1}{q_i^2\left(x_{i+1} - \frac{q_{i-1}}{q_i}\right)\left(a_{i+1} - \frac{q_{i-2}}{q_i}\right)} \cdot \frac{-1}{x_{i+2}}. \tag{52}$$

Now

$$\text{sgn.}\left(a_{i+1} - \frac{q_{i-1}}{q_i}\right) = \text{sgn.}\left(x_{i+1} - \frac{q_{i-1}}{q_i}\right)$$

and, consequently, the first factor in the second member of equation (52) is positive. Hence by replacing $i + 1$ by i, equation (52) may be written in the form

$$x_0 - \frac{p_i}{q_i} = \frac{-A_i}{x_{i+1}}, \tag{53}$$

where A_i is a positive number. Therefore

THEOREM II. *When* $x_{i+1} < 0$, $x_0 > \frac{p_i}{q_i}$ *and when* $x_{i+1} > 0$, $x_0 < \frac{p_i}{q_i}$.

It is now to be shown that

$$\left|x_0 - \frac{p_i}{q_i}\right| > \left|x_0 - \frac{p_{i+1}}{q_{i+1}}\right|. \tag{54}$$

By equation (48)

$$x_0 - \frac{p_{i+1}}{q_{i+1}} = \frac{-1}{q_{i+1}^2\left(x_{i+1} - \frac{q_i}{q_{i+1}}\right)}. \tag{55}$$

Now, on the other hand,

$$q_i^2\left(x_{i+1}-\frac{q_{i-1}}{q_i}\right)=q_i^2\left(a_{i+1}-\frac{1}{x_{i+2}}-\frac{q_{i-1}}{q_i}\right)=q_i\left(a_{i+1}q_i-q_{i-1}-\frac{q_i}{x_{i+2}}\right)$$
$$=\frac{q_i q_{i+1}\left(x_{i+2}-\dfrac{q_i}{q_{i+1}}\right)}{x_{i+2}}. \tag{56}$$

Hence

$$x_0-\frac{p_i}{q_i}=\frac{-x_{i+2}}{q_i q_{i+1}\left(x_{i+2}-\dfrac{q_i}{q_{i+1}}\right)}. \tag{57}$$

Since $|x_{i+2}|>1$ and $|q_{i+1}|\geqq|q_i|$

$$\left|\frac{-x_{i+2}}{q_i q_{i+1}\left(x_{i+2}-\dfrac{q_i}{q_{i+1}}\right)}\right|>\left|\frac{-1}{q_{i+1}^2\left(x_{i+2}-\dfrac{q_i}{q_{i+1}}\right)}\right|. \tag{58}$$

By virtue of equation (55) this establishes unequality (54). Hence

THEOREM III. *In the continued fraction*

$$\lambda\,(x_0)_{\lambda'}=(a_0, a_1, \ldots, a_{n-1}, x_n)$$

$\frac{p_{i+1}}{q_{i+1}}$ *is a closer approximation to* x_0 *than is* $\frac{p_i}{q_i}$, $0\leqq i<n$.

In $\lambda\,(x_0)_{\lambda'}$ we shall get a relatively close approximation to x_0 by stopping with the element a_{i+1} when both x_{i+1} and x_{i+2} are relatively great numerically with respect to a_i.

By equation (48) the error made in replacing x_0 by a convergent of a definite order may be determined; by inequality (54) it follows that the higher the order of the convergent the closer the approximation to x_0; and from the convergency of λ-developments ($\lambda=\lambda'$) that the approximation to x_0 may be made as close as desired by a convergent of a sufficiently high order. λ-developments ($\lambda=\lambda$) are, therefore, well adapted to approximate calculations.

§5. *Reverse λ-Developments.*

1.° The treatment of certain questions in the theory of binary quadratic forms and, in particular, those concerning λ-developments ($\lambda=\lambda'$) representing the square roots of integers not perfect squares is simplified when based on a system of continued fractions permitting reversal. The property of reversal belongs to simple continued fractions, to H_I and with a slight qualification to H_{II}. It is proposed to examine to what extent this property prevails in the system here under consideration. In particular it is desirable to specify what, if any, values of λ' exist such that for every λ-development $(\lambda = \lambda')$ there is a reverse λ-development for at least one value of λ, say λ'', independent of the particular λ-development in question. It will be found that the λ-developments $(\lambda = \lambda'$, $\lambda' = 1)$, added to H_I and H_{II} completes the list of such developments.

2.° Equation (2), §4, may be written in the form

$$\frac{q_{i+1}}{q_i} = a_{i+1} - \frac{q_{i-1}}{q_i}, \quad i = 0, 1, \ldots, n-2. \tag{1}$$

Adopting the notation $Q_i = \frac{q_i}{q_{i-1}}$, equation (1) becomes

$$Q_{i+1} = a_{i+1} - \frac{1}{Q_i}, \quad i = 0, 1, 2, \ldots, n-1. \tag{2}$$

Let i take for the moment a fixed value n_1. Then equation (2) becomes

$$Q_{n_1+1} = a_{n_1+1} - \frac{1}{Q_{n_1}}, \qquad 0 \leqq n_1 < n-1. \tag{3}$$

From the equations derived from equation (2) letting i take successively the values $0, 1, 2, \ldots, n_1$ eliminate $Q_2, Q_3, \ldots, Q_{n_1}$. The result may be expressed thus:

$$Q_{n_1+1} = (a_{n_1+1}, a_{n_1}, \ldots, a_2, a_1). \tag{4}$$

The second member of this equation is called a reverse development to the λ-development $(\lambda = \lambda')$ of x_0, where

$$\lambda\,(x_0)_{\lambda'} = (a_0, a_1, \ldots, a_{n-1}, x_n), \qquad n > n_1 + 1, \tag{5}$$

or, simply, a reverse development. When the second member of equation (4) satisfies the conditions under theorem II, §3, for $\lambda = \lambda''$, it will be referred to as a reverse λ-development $(\lambda = \lambda'')$.

3.° The necessary and sufficient condition that the second member of equation (4) be a λ-development $(\lambda = \lambda'')$ is, of course given by theorem II, §3; but for our present purpose it will be convenient to derive in another form a necessary condition.

When sgn. $a_{i+1} =$ sgn. a_i, then by equation (2)

$$|Q_{i+1}| < |a_{i+1}|;$$

and, when sgn. $a_{i+1} = -$ sgn. a_i,

$$|Q_{i+1}| > |a_{i+1}|, \qquad 0 < i \leqq n_1.$$

When $i = 0$, $Q_1 = a_1$.

Denote the maximum absolute value of $\frac{1}{Q_{i+1}}$, $0 < i \leqq n_1$, when sgn. $a_{i+1} =$ sgn. a_i by $g_{\lambda'}$, and the like maximum of $\frac{1}{Q_{i+1}}$, $0 < i \leqq n_1$, when sgn. $a_{i+1} = -$ sgn. a_i by $g'_{\lambda'}$. Now since the extent of interval associated with any non-zero integer is at most unity and further since all normal unit intervals have the same extent on the zero side of their representative integers and, again, the same extent on the side toward infinity, in order that there exist a value of λ, say λ'_1, for which the second member of equation (4) is a λ-development it is necessary that

$$g_{\lambda'} + g'_{\lambda} \leqq 1. \tag{6}$$

This necessary condition being satisfied, does there exist a value of λ_1, say λ'', for which $\lambda\,(x_0)_{\lambda'}$ has a reverse λ-development ($\lambda = \lambda''$) for every value of x_0? The necessary criterion by which to answer this question may be derived from equation (6). Let x_0 run through the whole range of real values and for every such value of x_0 from the corresponding $g_{\lambda'}$ and $g'_{\lambda'}$. Let $\gamma_{\lambda'}$ denote the maximum value or the maximum limit of the $g_{\lambda'}^{\prime s}$, and $\gamma'_{\lambda'}$, the maximum value or the maximum limit of the $g_{\lambda'}^{\prime s}$.

Then the necessary condition for the existence of a value of λ_1, say λ'', for which for every x_0 $\lambda\,(x_0)_{\lambda'}$ has a reverse λ-development ($\lambda = \lambda''$) is

$$\gamma_{\lambda'} + \gamma'_{\lambda'} \leqq 1. \tag{7}$$

4.° The values of $g_{\lambda'}$ and $g'_{\lambda'}$ are to be derived from the minimum values or minimum limits of $|Q_i|$, $i = 1, 2, \ldots, n_1 - 1$, in the particular $\lambda\,(x_0)_{\lambda'}$ under consideration.

Replacing in equation (2) i by $i - 1$ we have

$$Q_i = a_i - \frac{1}{Q_{i-1}}. \tag{8}$$

Under the hypothesis that sgn. $a_i =$ sgn. a_{i-1}, $|Q_i|$ is a minimum when and only when $|a_i|$ and $|Q_{i-1}|$ take the minimum values consistent with the laws of λ-developments ($\lambda = \lambda'$). The step by step application of this reasoning leads to the conclusion that $|Q_i|$ is a minimum when $a_1, a_2, \ldots, a_{i-1}, a_i$ all agree in sign and have the least absolute values permissible in λ-developments ($\lambda = \lambda'$). When $|a_i| \geqq |a_{i-1}| + 1$, then $|Q_i| \geqq |Q_{i-1}|$, $1 \leqq i < n_1$.

When sgn. $a_i = -$sgn. a_{i-1}, $|Q_i|$ is a minimum if $|a_i|$ is a minimum and $|Q_{i-1}|$ a maximum.

To apply test (6) note that

$$(a_0,\ 2,\ a_2,\ \ldots,\ a_k,\ 2,\ a_2,\ \ldots,\ a_k,\ 2,\ a_n), \quad n = 2\,k + 2, \tag{10}$$

where all the elements except a_n are positive is formally a λ-development ($\lambda = \lambda'$, $0 < \lambda' < \frac{1}{2}$). In this case

$$Q_i = (2,\ a_k,\ \ldots,\ a_2,\ 2), \quad i = k + 1, \tag{11}$$

and

$$g_{\lambda'} \geqq \frac{1}{2 - \frac{1}{a_k}}. \tag{12}$$

Again

$$Q_i = (2,\ a_k,\ \ldots,\ a_2,\ 2,\ a_k,\ \ldots,\ a_2,\ 2), \quad i = 2\,k + 1. \tag{13}$$

Since by hypothesis $a_n < 0$,

$$g'_{\lambda'} \geqq \frac{1}{2 - \frac{1}{a_k}}. \tag{14}$$

Therefore

$$g_{\lambda'} + g'_{\lambda'} \geqq \frac{2}{2 - \frac{1}{a_k}} > 1, \tag{15}$$

and the necessary condition (b_0) is not satisfied. Hence for every value of λ', $0 < \lambda' < \frac{1}{2}$, there exists values of x_0 such that for $\lambda\,(x_0)_{\lambda'}$ there is no value of λ for all positive values of n_1 and n,

$$Q_{n_1} = (a_{n_1},\ a_{n_1 - 1},\ \ldots,\ a_2,\ a_1), \quad 0 < n_1 < n, \tag{16}$$

is a reverse λ-development.

For $\frac{1}{2} < \lambda' < \tau$, $\tau \equiv \frac{-1 + \sqrt{5}}{2}$ consider the sequence

$$(a_0,\ \ldots,\ a_j,\ 2,\ a_{j+2},\ 3_1,\ 3_2,\ \ldots,\ 3_k,\ 2,\ a_n), \quad n = j + k + 4, \tag{17}$$

where $a_0 > 0$, $a_j < 0$, $a_{j+2} > 0$, $a_n < 0$, and properly chosen.

When choice has been made of a particular value for λ' then certainly the a's must fulfill certain conditions but which in no case involve a change of sign of the a's. To determine approximately $g_{\lambda'}$ let

$$Q_i = (2,\ a_i;\ \ldots,\ a_1)\ .\ i = j + 1.$$

Then, certainly,

$$2 + \frac{1}{|a_j| + 1} \leqq |Q_i| \leqq 2 + \frac{1}{|a_j| - 1}, \tag{18}$$

and, correspondingly,

$$\frac{1}{2 + \frac{1}{|a_j| + 1}} \geqq g_{\lambda'} \geqq \frac{1}{2 + \frac{1}{|a_j| - 1}} \tag{19}$$

Now when $j = 0$, or $a_j = \infty$

$$g_{\lambda'} = \tfrac{1}{2}. \tag{20}$$

To find approximately $g'_{\lambda'}$ set

$$Q_i = (2,\ 3_k,\ \ldots,\ 3_2,\ 3_1,\ a_{j+2},\ 2,\ a,\ \ldots\ a_2,\ a_1). \tag{21}$$

As k increases without limit the value of Q_i may approach as nearly as we please to

$$2 - \frac{2}{3 + \sqrt{5}} = \frac{1 + \sqrt{5}}{2}. \tag{22}$$

Since, by hypothesis, $a_n < 0$, the value of $g'_{\lambda'}$ may be made to approach as nearly as we please to the number $\dfrac{\sqrt{5} - 1}{2}$. Hence by a proper choice of i and k in sequence (17) the $g_{\lambda'}$ sum $g_{\lambda'} + g'_{\lambda'}$ may be made to approach as nearly as we please to the sum

$$\tfrac{1}{2} + \frac{\sqrt{5} - 1}{2} = \frac{\sqrt{5}}{2} > 1. \tag{23}$$

Hence for every value of λ', $\frac{1}{2} < \lambda' < \tau$, there exists an x_0 such that for $\lambda(x_0)_{\lambda'}$ there are no values of λ for which for all positive values of n_1 and n

$$Q_{n_1} = (a_{n_1},\ a_{n_1 - 1},\ \ldots,\ a_2,\ a_1),\ 0 < n_1 < n, \tag{24}$$

is a reverse λ-development.

When $\tau < \lambda' < 1$ the element $+1$ may enter a λ-development $(\lambda = \lambda')$ when immediately preceded and followed by negative elements. Accordingly, choose the λ-sequence $(\lambda = \lambda')$

$$(a_0,\ 1,\ a_2,\ a_3,\ a_4,\ a_5), \tag{25}$$

where the a's are negative. Without difficulty we find $g'_{\lambda'} = 1$ while $g_{\lambda'} > 0$. Hence

$$g_{\lambda'} + g'_{\lambda'} > 1, \tag{26}$$

and the necessary condition (6) is not satisfied. Hence for every $\lambda'_1 \tau < \lambda' < 1$, there exists an x_0 such that for $\lambda(x_0)_{\lambda'}$ there are no values of λ for which all positive values of n_1 and n

$$Q_{n_1} = (a_{n_1},\ a_{n_1 - 1},\ \ldots\ a_2,\ a_1)\quad 0 < n_1 < n, \tag{27}$$

is a reverse λ-development.

5.° When $\lambda' = \frac{1}{2}$, we may choose the λ-sequence

$$(a_0,\ 3_1,\ 3_2,\ \ldots,\ 3_i,\ 2_1\ a_{i+2}) \tag{28}$$

where $a_0 > 0$, $a_{i+2 < 0}$. To determine $g_{\lambda'}$ set

$$Q_i = (3_i,\ 3_{i-1},\ \ldots,\ 3). \tag{29}$$

Let $i = \infty$, then

$$Q_i = \frac{3+\sqrt{5}}{2} \tag{30}$$

and

$$g_{\lambda'} = \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}. \tag{31}$$

To determine $g'_{\lambda'}$ set

$$Q_{i+1} = (2_1\, 3_i,\, 3_{i-1},\, \ldots,\, 3_1) \tag{32}$$

and let i approach infinity. Then

$$Q_{i+1} = 2 - \frac{2}{3+\sqrt{5}} = \frac{1+\sqrt{5}}{2}. \tag{33}$$

Hence

$$g'_{\lambda'} = \frac{-1+\sqrt{5}}{2}. \tag{34}$$

Therefore

$$g'_{\lambda'} + g'_{\lambda'} = 1. \tag{35}$$

The values given of $g_{\lambda'}$ and $g'_{\lambda'}$ are respectively the values of γ_λ and $\gamma'_{\lambda'}$. For it follows by Theorem I, §3, for $\lambda' = \frac{1}{2}$ that sgn. $a_{i+1} = -$ sgn. a_i when $a_i = 2$. Hence the sequence (29) gives the absolutely least value of Q_{i+1} for sgn. $a_i =$ sgn. a_{i-1}; also sequence (32) gives the least value of Q_{i+1} when sgn. $a_{i+1} = -$ sgn. a_i. In this case the value of λ'' is $g'_{\lambda'} = \dfrac{-1+\sqrt{5}}{2} \equiv \tau$.

The necessary condition is also sufficient. For, when $\lambda = \lambda'' = \tau$, the end intervals are also normal unit intervals and consequently, $Q_{i+1} \sim a_{i+1}$, $i = 0, 1, 2, \ldots n_1$. Further, the conditions imposed on the elements a by $\lambda = \lambda' = \frac{1}{2}$ is precisely the same as those imposed on them by $\lambda = \lambda'' = \tau$ with this exception that for $\lambda' = \frac{1}{2}$. $a = \pm 2$ is always followed by a change of sign, while for $\lambda'' = \tau_1\, a = \pm 2$ is always preceded by a change of sign. Hence when the order of the elements a for $\lambda' = \frac{1}{2}$ is reversed these elements in the reversed order satisfy the conditions imposed when $\lambda = \lambda'' = \tau$.

For $\lambda = \tau_1$ choose the λ-sequence ($\lambda = \lambda' = \tau$)

$$a_0,\, 2,\, a_2,\, -2,\, a_4,\, \ldots,\, a_j), \tag{36}$$

where $a_0 < 0$, $a_2 > 0$, $a_4 > 0$.

Then

$$Q_i = 2 \tag{37}$$

and

$$g_{\lambda'} = \gamma_{\lambda'} = \frac{1}{2}. \tag{38}$$

Again

$$Q_3 = (-2,\, a_2,\, 2) \tag{39}$$

and when $a_t = \infty$, $Q_3 = -2$. Hence

$$g_{\lambda'} = \tfrac{1}{2}. \tag{40}$$

It may be shown that $\gamma'_{\lambda'} = g'_{\lambda'}$. Therefore by equations (38) and (40)

$$\gamma'_{\lambda'} + \gamma'_{\lambda'} = 1. \tag{41}$$

Here $\lambda = \lambda'' = \gamma'_{\lambda'} = \frac{1}{2}$.

The necessary condition is in this case insufficient. For example consider the λ-sequence ($\lambda = \tau$)

$$(a_0, -2, a_2, \ldots, a_n, a_{n+1}), \; a_2 > 0 \tag{42}$$

and from the reverse development

$$Q_{n+1} = (a_{n+1}, a_n, \ldots, a_2, -2). \tag{43}$$

This is not a λ-development ($\lambda \equiv \lambda' = \frac{1}{2}$), since by theorem I, 3°, of §3, $\lambda\,(x_0)_{\lambda'}$, cannot terminate with η_1 and in this case $\eta_1 = -2$, the last element in the continued fraction (43). With the exception of such reverse developments as end with the element $\pm\,\mathbf{2}$ preceded by an element with the opposite sign, the necessary condition is also sufficient.

In all cases, therefore, except when in $\lambda\,(x_0)_{\lambda'=\tau}$ $|a_1| = 2$, while sgn. $a_t =$ $-$ sgn. a_1, there is for every $\lambda\,(x_0)_{\lambda'=\tau}$, x_0 real, a reverse λ-development ($\lambda = \lambda'' = {}^1$) for every positive n_1 and n, $0 < n_1 < n$.

When $\lambda' = 1$, $g'_{\lambda'} = \gamma'_{\lambda} = 1$ and $g_{\lambda'} = \gamma_{\lambda'} = 0$. The necessary condition for reverse λ-developments ($\lambda = \lambda'_1 = 1$) is satisfied. An exception, however, must be made as in the last case. The last element in the reverse λ-development ($\lambda = \lambda'' = 1$) must not be $\pm\,1$.

The results obtained may be summed up in the following theorem:

Theorem I. *All λ-developments ($\lambda = \lambda' = \frac{1}{2}$) have reverse λ-developments ($\lambda = \lambda'' = \tau$)*; all those and only those λ-developments ($\lambda = \lambda' = \tau$) in which the absolute value a_1 is not 2 when sgn. $a_1 = -$ sgn. a_1 have reverse λ-developments ($\lambda = \lambda'' = \frac{1}{2}$)*; also, all those and only those λ-developments ($\lambda = \lambda' = 1$) in which the absolute value a_1 is not 1 have reverse λ-developments ($\lambda = \lambda'' = 1$).*

6.° The following theorems deal with special cases of reverse λ-developments.

Theorem II. *Every λ-development ($\lambda = \lambda'$, $0 < \lambda' \leqq 1$) into which neither $\pm\,1$ nor $\pm\,2$ enters as an element after the first has a reverse λ-development ($\lambda = \lambda' = \tau$).*

For every such development is a λ-development ($\lambda = \lambda' = \frac{1}{2}$).

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* These results are due to Hurwitz, see *Acta Math.* 12, pp. 379, 380.

THEOREM III. *Every λ-development ($\lambda = \lambda'$, $0 < \lambda' \leqq 1$) which after the first element has only positive or only negative elements has a reverse λ-development.*

For, in this case, $\gamma_{\lambda'} < 1$ and $\gamma'_{\lambda'} = 0$. Hence the reverse development is a λ-development ($\lambda = \lambda''$) where

$$\lambda'' = (0) \dots . 1 - \gamma_{\lambda'}.$$

THEOREM IV. *Every λ-development ($\lambda = \lambda'$, $0 < \lambda' \leqq \frac{1}{2}$) in which, the first element aside, enter at least two consecutive twos with like signs as elements followed immediately by a change of sign has no reverse λ-development.*

For definiteness let at least one set of twos be positive. Then by the general theory of λ-developments ($\lambda = \lambda'$) the element preceding the first two of the set is positive. Hence $Q_i \leqq 2$, where i is the rank of the first two of the set. Hence

$$g_{\lambda'} \geqq \tfrac{1}{2}. \tag{44}$$

Let i' be the rank of the last two of the set. Then $Q_{i'} \leqq \frac{3}{2}$. Hence

$$g'_{\lambda'} \geqq \tfrac{2}{3}; \tag{45}$$

and, consequently,

$$g_{\lambda'} + g'_{\lambda'} > 1. \tag{46}$$

Therefore, the necessary condition (6) for a reverse λ-development to the λ-developments of the description set forth in the theorem is not satisfied.

The two following theorems for very special cases are appended without demonstration.

THEOREM V. *Every λ-development ($\lambda = \lambda'$, λ' the reciprocal of an integer equal to or greater than 2) in which every element just preceding a change of sign is in absolute value equal to or greater than $\frac{1}{\lambda'} + <$ has a reverse λ-development.*

THEOREM VI. *Every λ-development ($\lambda = \lambda'$, λ' the reciprocal of an integer equal to or greater than 2) not ending with a maximum set of twos with like signs has a reverse λ-development provided that whenever, the first element aside, there occurs a sequence of which each element and the following have unlike signs, the first element in this sequence is in absolute value equal to or greater than $\frac{1}{\lambda'} + 1$.*

§6. *A Comparison of Certain Types of Continued Fraction Developments.*

1.° Since the results of this section are derived without difficulty and have no direct bearing on what follows they are given without proof.

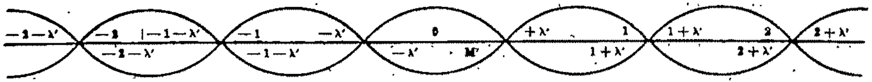
Consider the two systems of intervals, viz.

I. $\ldots\,;\ (-2-\lambda')\ldots\ldots -1-\lambda';\ (-1-\lambda')\ldots\ldots -\lambda';\ (-\lambda')\ldots\ldots(+\lambda);$
$\lambda'\ldots\ldots(1+\lambda);\ 1+\lambda'\ldots\ldots(2+\lambda');\ \ldots$

and

II. $\ldots\,;\ 2-\lambda'\ldots\ldots(-1-\lambda');\ -1-\lambda'\ldots\ldots(-\lambda');\ -\lambda'\ldots\ldots+\lambda';$
$(\lambda')\ldots\ldots 1+\lambda';\ (1+\lambda')\ldots\ldots 2+\lambda';\ \ldots$

These systems may be graphically represented as in the figure below by the intervals above and below the lines, the interval limit being written in every instance in the interval to which it belongs. These two systems will be referred to as interval system I, and interval system II, respectively.



Consider also the two types of defining equations, viz.,

$$x_i = a_i - \frac{1}{x_{i+1}}, \qquad i = 0,\ 1,\ 2,\ \ldots, \tag{1}$$

$$x_i = a_i + \frac{1}{x_{i+1}}, \qquad i = 0,\ 1,\ 2,\ \ldots. \tag{2}$$

The two types of interval systems taken in connection with the two types of defining equations give rise to four types of continued fractions. For brevity we shall write λ-developments $(\lambda = \lambda';\ I_t)$, $t = 1$ or 2, for continued fractions based on the interval system I and defining equations (1) or (2) according as $t = 1$ or 2. In an analogous manner the notation λ-developments $(\lambda = \lambda';\ II_t)$ is to be interpreted. When a particular number x_0 is under consideration we may write $\lambda(x_0)_{\lambda';\,I}$, in $\lambda(x_0)_{\lambda'}$ as hitherto, to avoid ambiguity.

2.° Continued fractions based on defining equation (1).

Theorem I. *Only for those numbers whose λ-developments $(\lambda = \lambda';\ I_1)$ from an element of a certain rank on coincide with* $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda';\,I_1}$, *or with* $\lambda\left(-\frac{1}{1-\lambda'}\right)_{\lambda';\,I_1}$, *do the λ-developments $(\lambda = \lambda';\ I_1)$ differ from the corresponding λ-developments $(\lambda = \lambda';\ II_1)$.*

Theorem II. *The numbers whose λ-developments $(\lambda = \lambda';\ I_1)$ from an element of a certain rank on are identical with* $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda';\,I_1}$, *or with* $\lambda\left(-\frac{1}{1-\lambda'}\right)_{\lambda';\,I_1}$, *have* λ-*developments* $(\lambda = \lambda'; II_1)$ *identical from an element of a certain rank on with* $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda';\, II_1}$, *or with* $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda';\, II_1}$.

3.° Continued fractions based on defining equation (2).

THEOREM III. *Only for those numbers whose* λ-*developments* $(\lambda = \lambda'; I_2)$ *from an element of a certain rank on are identical with* $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda';\, I_2}$ *or* $\lambda\left(-\frac{1}{1-\lambda'}\right)_{\lambda';\, I_3}$, *do the* λ-*developments* $(\lambda = \lambda'; I_2)$ *differ from the corresponding* λ-*developments* $(\lambda = \lambda'; II_2)$.

THEOREM IV. *The number whose* λ-*developments* $(\lambda = \lambda'; I_2)$ *from an element of a certain rank on are identical with* $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda';\, I_2}$, *or with* $\lambda\left(-\frac{1}{1-\lambda'}\right)_{\lambda';\, I_2}$. *have* λ-*developments* $(\lambda = \lambda', II_2)$ *identical from an element of a certain rank on with* $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda';\, II_2}$, *or with* $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda';\, II_2}$.

4.° Comparison of continued fractions arising from defining equations (1) and (2).

THEOREM V. *When for either interval system and for* $\lambda = \lambda'$ x_0 *is expressed as a continued fraction based on defining equation* (1), *this continued fraction differs from that of* x_0 *in the same interval system and for the same* λ, $\lambda = \lambda'$, *but based on defining equation* (2) *only in this: the elements of odd rank in the first continued fraction appear with opposite signs in the second continued fraction.*

REMARK I. The continued fraction development of a positive number for $\lambda' = 1$ and based on defining equation (2) and interval system I_1 is identically the simple continued fraction representing the number.

REMARK II. The periodicity of continued fractions $(\lambda = \lambda')$ representing quadratic irrationalities when these continued fractions belong to one of the types considered in this section implies in the remaining types the periodicity of the continued fractions representing quadratic irrationalities.

§7. *Equivalence.*

1.° In simple continued fractions the equivalence, proper or improper, of two real numbers appears in the ultimate likeness* of the continued fractions representing the numbers. In λ-developments $(\lambda = \lambda')$ a distinction is made between proper and improper equivalence except when the two numbers in question are both properly and improperly equivalent. The relation of the λ-developments ($\lambda = \lambda'$) of two properly or improperly equivalent numbers is further dependent on the particular value λ' assigned to λ.

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* To avoid a special case it is here assumed that the last element in the continued fraction representing a rational number ∞.

The λ-developments ($\lambda = \lambda', \tau < \lambda' \leqq 1$) of two properly equivalent numbers are ultimately alike; those of two improperly equivalent numbers differ ultimately in the signs of the elements and in this respect only.

For λ-developments ($\lambda = \lambda'$; $0 < \lambda' \leqq \tau$) two cases arise. Let x_0' be properly equivalent to x_0. Then a) $\lambda\,(x_0')_{\lambda'}$ is ultimately like $\lambda\,(x_0)_{\lambda'}$; or b) $\lambda\,(x_0')_{\lambda'}$ is ultimately like a λ-development ($\lambda = \lambda'$) derived by a simple rule (to be explained hereafter) from $\lambda\,(x_0)_{\lambda'}$ and called the *associate* of $\lambda\,(x_0)_{\lambda'}$. To cover the case of the improper equivalence of x_0' and x_0 we need in the preceding statement merely replace $\lambda\,(x_0)_{\lambda'}$ by $\lambda\,(-x_0)_{\lambda'}$.

A very special case of b) appears in H_I and H_{II}[*] but hardly in such a way as to make its real significance obvious. In the present general theory b) assumes an important role and shows in a particularly interesting way how the relation of $\lambda\,(x_0')_{\lambda=\lambda'}$ to $\lambda\,(x_0)_{\lambda=\lambda}$ depends on the relation of the values of the ultimate elements in $\lambda\,(x_0)_{\lambda=\lambda'}$ to the particular value of λ' assigned to λ.

2.° Two real numbers x_0 and x_0' are said to be properly equivalent when they satisfy an equation of the form

$$x_0' = \frac{\alpha x_0 - \beta}{\gamma x_0 - \delta}. \tag{1}$$

in which α, β, γ, δ, are integers such that

$$\beta\gamma - \alpha\delta = 1. \tag{2}$$

Theorem I.[†] *When in the λ-developments of two real numbers x_0 and x_0' for the same or for different values of λ; viz., in*

$$\lambda\,(x_0) = (a_0, a_1, \ldots, a_{n-1}, x_n),$$
$$\lambda\,(x_0') = (a_0', a_1', \ldots, a_{m-1}, x_m),$$

$x_m = x_n$ for some values of m and n, then x_0 and x_0' are properly equivalent.

For, in the usual notation

$$x_0 = \begin{pmatrix} p, & -q \\ r, & -s \end{pmatrix} x_n, \tag{3}$$

$$x_0' = \begin{pmatrix} p', & -q' \\ r', & -s' \end{pmatrix} x_m, \tag{4}$$

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[*] Hurwitz: *loc. cit.*

[†] Cf. Hurwitz: *loc. cit.*

where $p, q, r, s, p', q', r', s'$ are integers satisfying the relation

$$\begin{vmatrix} p, & -q \\ r, & -s \end{vmatrix} = \begin{vmatrix} p', & -q' \\ r', & -s' \end{vmatrix} = 1.$$

From equations (3) and (4), since $x'_m = x_n$

$$x'_0 = \begin{pmatrix} \alpha, & -\beta \\ \gamma, & -\delta \end{pmatrix} x_0, \tag{5}$$

where $\alpha, \beta, \gamma, \delta$ are integers depending on $p, q, r, s, p', q', r', s'$ and satisfying equation (2).

3.° Given that x_0 and x'_0, two real numbers, are properly equivalent, we seek to determine the relation of their λ-developments ($\lambda = \lambda'$). In what follows x_0 and x'_0 are by hypothesis irrational; the results become applicable to rational numbers on adopting the convention that every element or partial denominator in their λ-developments ($\lambda = \lambda'$) from those of a certain rank on is infinite.

When x_0 and x'_0 are related as in equation (1) we shall say that x_0 passes into x'_0 by the proper unitary substitution

$$\begin{pmatrix} \alpha_1 - \beta \\ \gamma_1 - \gamma \end{pmatrix}.$$

Let S and T be defined by the following equations:

$$S_{(x)} \equiv x + 1, \qquad T_{(x)} \equiv -\frac{1}{x}. \tag{6}$$

Then S and T denote proper unitary substitution and are called elementary substitutions.

The theory of equivalence here developed is based on the well-known proposition.*

THEOREM II. *Every proper unitary substitution is a product of the elementary substitutions S and T.*

4.° We shall determine directly the transformations which $\lambda(x_0)_{\lambda'}$ undergoes when S and T separately operate on x_0, and from these results derive by means of theorem II the changes produced in $\lambda(x_0)_{\lambda'}$ when x_0 is operated on by any proper unitary substitution.

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* Cf. Mathews: Theory of Numbers, Part I, Art. 112; or Jordan: Cours d'analyse, 2d. Edn., Vol. II, pp. 327–330; also Klein: Theorie der elliptischen Modulfunktionen, I, pp. 218, 219.

The theory of equivalence was worked out following the method of Hurwitz. The present method as probably offering a simpler solution was taken up at the suggestion of Prof. E. H. Moore.

Theorem III. *For x_0 real $\lambda(x_0)_{\lambda'}$ and $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ from elements of a certain rank on are identical except when $\lambda' < |x_0| < \frac{1}{\lambda'}$, $0 < \lambda' \leqq \tau$, and the complete quotients beginning with x_1 all have the same sign and are in absolute value equal to or less than $\frac{1}{\lambda'} + 1$.*

Let it be assumed that

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{n-1}, x_n). \tag{7}$$

Then

$$x_0 = a_0 - \frac{1}{x_1} = \frac{a_0 x_1 - 1}{x_1} \tag{8}$$

and

$$-\frac{1}{x_0} = \frac{x_1}{1 - a_0 x_1}. \tag{9}$$

There are several cases to consider.

$$\text{a)} \quad |x_0| > \frac{1}{\lambda'}.$$

In this case

$$-\frac{1}{x_0} \sim 0 \tag{10}$$

and

$$-\frac{1}{x_0} = 0 - \frac{1}{a_0 - \frac{1}{x_1}} = (0, a_0, x_1). \tag{11}$$

Therefore

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = (0, a_0, a_1, \dots, a_{n-1}, x_n). \tag{12}$$

$$\text{b)} \quad |x_0| < \lambda'.$$

In this case

$$x_0 \sim 0 \tag{13}$$

and

$$-\frac{1}{x_0} = x_1. \tag{14}$$

Therefore

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = (a_1, a_2, \dots, a_{n-1} x_n) \tag{15}$$

$$\text{c)} \quad |x_0| = \lambda', \; |x_0| = \frac{1}{\lambda'}.$$

These cases have been dealt with fully in § 2. The results were entirely in cord with theorem III.

$$\text{d)}\quad \lambda' < |x_0| < \frac{1}{\lambda'}.$$

It is sufficient to consider the case in which $x_0 > 0$. The result for $x_0 < 0$ n be obtained at once from the symmetry of the interval system

$$\text{d}_1)\quad a_0 = 1.$$

Then

$$x_0 = 1 - \frac{1}{x_1} \tag{16}$$

ıd

$$-\frac{1}{x_0} = -1 - \frac{1}{x_1 - 1}. \tag{17}$$

hen $-\frac{1}{x_0} \sim -1$, as is always true for $\tau < \lambda' \overset{=}{<} 1$, then

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = (-1,\, x_1 - 1) = (-1,\, a_1 - 1,\, a_2,\, \ldots,\, a_{n-1},\, x_n). \tag{18}$$

ıd

$$x_1 = \left(\frac{1}{\lambda'} + 1\right) \cdots\cdots -\frac{\lambda'}{1-\lambda'}.$$

nce the normal interval for x_1 is

$$x_1 = \frac{1}{1-\lambda'} \cdots\cdots \left(-\frac{1}{\lambda'}\right),$$

ere remains for consideration the two cases:

$$x_1 = \left(-\frac{\lambda'}{1-\lambda'}\right) \cdots\cdots \left(-\frac{1}{\lambda'}\right), \quad x_1 < 0. \tag{19}$$

ıd

$$x_1 = \frac{1}{1-\lambda'} \cdots\cdots \frac{1}{\lambda'} + 1. \tag{20}$$

The first of this, (19), implies that $\tau < \lambda' \overset{=}{<} 1$ and is consequently included in ıe case $-\frac{1}{x_0} \sim -1$. For the second case, (20),

$$\frac{1}{1-\lambda'} \overset{=}{<} x_1 \overset{=}{<} \frac{1}{\lambda'} + 1$$

follows that $0 < \lambda' \overset{=}{<} \tau$. The complete treatment of this case is deferred until preliminary examination has been made of the other cases.

$$\text{d}_2)\quad x_0 > 0, \quad a_0 > 1.$$

Without at all limiting the generality of this case it may be assumed that

$$a_1 = a_2 = \ldots\ldots = a_h = 2 \neq a_{h+1},\ h \text{ a positive integer.}$$

In this case*

$$\frac{1}{x_0} \sim 1. \tag{21}$$

Now

$$-\frac{1}{x_0} \equiv -1 + \frac{x_0 - 1}{x_0} \equiv -1 + \cfrac{1}{1 + \cfrac{1}{x_0 - 1}}. \tag{22}$$

Hence

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = \left(-1, -1 - \frac{1}{x_0 - 1}\right). \tag{23}$$

Since $x_0 < \frac{1}{\lambda'}$ then

$$1 + \frac{1}{x_0 - 1} > 1 + \lambda'$$

and in consequence $1 + \frac{1}{x_0 - 1}$ corresponds to 2 or to some integer greater than 2. In this case we may say that

$$\lambda\left(1 + \frac{1}{x_0 - 1}\right)_{\lambda'} = \left(2_1, 2_2, \ldots\ldots, 2_{a_0 - 2}, 2 + \frac{1}{x_1 - 1}\right). \tag{24}$$

To prove this it is necessary to show that

$$2_i - \frac{x_0 - i - 1}{x_0 - i} \sim 2_i,\ i = 1, 2, \ldots\ldots, a_0 - 2. \tag{25}$$

Now

$$2_i - \frac{x_0 - i - 1}{x_0 - i} = 1 + \frac{1}{x_0 - i}. \tag{26}$$

It is to be shown, therefore, that

$$1 + \frac{1}{x_0 - i} = 1 + \lambda' \ldots\ldots (2 + \lambda'),\ i = 1, 2, \ldots\ldots, a_0 - 2. \tag{27}$$

Since

$$x_0 < \frac{1}{\lambda'}$$

then, certainly,

$$0 < x_0 - i < \frac{1}{\lambda'},\quad 1 \leqq i \leqq a_0 - 2,$$

and, consequently,

$$1 + \frac{1}{x_0 - i} > 1 + \lambda'. \tag{28}$$

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*When $\tau \leqq \lambda' \leqq 1$, $h = 0$. But all cases in which $\tau < \lambda' \leqq 1$ are disposed of completely under d$_1$). We may say that d$_2$) includes d$_1$) under the hypothesis that $h = 0$, and removing the restriction on a_0.

On the other hand

$$x_0 - i \overline{\underset{>}{}} x_0 - a_0 + 2 = 1 + \lambda' \ldots (2 + \lambda'). \tag{29}$$

Hence

$$\frac{1}{x_0 - i} < 1.$$

and

$$1 + \frac{1}{x_0 - i} < 2 < 2 + \lambda'. \tag{30}$$

By the inequalities (28) and (30) follow the interval equation (27) and thence relation (25).

Hence, by successive reduction, we get

$$\lambda\left(1 + \frac{1}{x_0 - 1}\right)_{\lambda'} = \left(2_1, 2_2, \ldots, 2_{a_0-2}, \frac{x_0 - a_0 + 2}{x_0 - a_0 + 1}\right). \tag{31}$$

By virtue of the relation

$$x_0 - a_0 = -\frac{1}{x_1}$$

equation (31) reduces to equation (24). Combining equations (23) and (24)

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = \left(-1, -2_1, -2_2, \ldots, -2_{a_0-2}, -2 - \frac{1}{x_1 - 1}\right). \tag{32}$$

When

$$-2 - \frac{1}{x_1 - 1} \sim -2,$$

then

$$\begin{aligned} \lambda\left(-\frac{1}{x_0}\right)_{\lambda'} &= (-1, -2_1, -2_2, \ldots, -2_{a_0-2}, -2_{a_0-1}, x_1 - 1) \\ &= (-1, -2_1, -2_2, \ldots, -2_{a_0-2}, -2_{a_0-1}, a_1 - 1, a_2, \ldots, a_{n-1}, x_n). \end{aligned} \tag{33}$$

When $-2 - \dfrac{1}{x_1 - 1}$ does not correspond to -2, a further reduction is necessary.

$$d_3) \quad x_0 > 0, \ a_0 \overline{\underset{>}{}} 1, \ 0, < \lambda' \overline{\underset{<}{}} \tau.$$

To include in one investigation the deferred case under d_1) and that under d_2 we may study $\lambda\left(a + \dfrac{1}{x_1 - 1}\right)_{\lambda'}$, $a > 0$, when $a + \dfrac{1}{x_1 - 1}$ does not correspond to a. By means of the symmetry of the interval system we then derive at once $\lambda\left(-a - \dfrac{1}{x_1 - 1}\right)_{\lambda'}$. When $a = 1$, this gives the deferred case under d_1) and when $a = 2$, that under d_2).

Observe first that $a+\frac{1}{x_1-1}$ does not correspond to $a-1$. For when $a=1$ this would make zero an element following the first in $\lambda\left(-\frac{1}{x_0}\right)$, which is not possible. When $a=2$ we should have 1 as an element not the first in $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ and hence $\tau<\lambda'\leqq 1$. But the case for $\tau<\lambda'\leqq 1$ was completely disposed of under d$_1$). Hence we study $\lambda\left(a+\frac{1}{x_1-1}\right)_{\lambda'}$, $0<\lambda'\leqq\tau$, for those cases in which $a+\frac{1}{x_1-1}\sim a+1$ or to some greater integer.

In this case
$$a+\frac{1}{x_1-1}\geqq a+\lambda', \qquad 0<\lambda'\leqq\tau, \tag{34}$$
and in consequence
$$0<x_1\leqq\frac{1}{\lambda'}+1. \tag{35}$$
Hence sgn. $x_1=$ sgn. $a_0=+1$. Therefore the condition (35) on x_1 may be replaced by
$$\frac{1}{1-\lambda'}\leqq x_1\leqq\frac{1}{\lambda'}+1. \tag{36}$$

It is now assumed for both deferred cases under d$_1$) and d$_3$) that
$$a_1=a_2=\ldots\ldots=a_h=2\neq a_{h+1}, \tag{37}$$
h a positive integer. On this hypothesis
$$x_i=2-\frac{1}{x_{i+1}} \qquad i=1,\ 2,\ \ldots\ldots,h \tag{38}$$
and, consequently,
$$\frac{1}{1-\lambda'}\leqq x_i\leqq\frac{1}{\lambda'}+1, \qquad i=1.\ 2,\ \ldots\ldots,h. \tag{39}$$

By equation (38) it is evident that
$$a+\frac{1}{x_1-1}=a+1+\frac{1}{x_2-1}=a+2+\frac{1}{x_3-1}=\ \ldots\ldots=a+h+\frac{1}{x_{h+1}-1}. \tag{40}$$

The further reduction of this expression will be made in two subcases.

$$\text{d}_{3_1})\quad a+h+\frac{1}{x_{h+1}-1}\sim a+h.$$

Then
$$\lambda\left(a+\frac{1}{x_1-1}\right)_{\lambda'\,(0<\lambda'\leqq\tau)}=(a+h_1-x_{h+1}+1)=(a+h,-a_{h+1}+1, \\ -a_{h+2},\ \ldots\ldots,-a_{h-1},-x_n). \tag{41}$$

$$d_{3_2})\quad a+h+\frac{1}{x_{h+1}-1}\text{ does not correspond to }a+h.$$

In this case $a+h+\dfrac{1}{x_{h+1}-1}$ does not correspond to $a+h-1$ or to a lesser integer. For, did the correspondence hold, then

$$a+h+\frac{1}{x_{h+1}-1}<a+h-1+\lambda', \tag{42}$$

whence it follows that $x_{h+1}<0$ and further

$$|x_{h+1}|>\frac{1}{\lambda'}. \tag{43}$$

Hence

$$a+h+\frac{1}{x_{h+1}-1}>a+h-\frac{\lambda'}{1+\lambda'}. \tag{44}$$

Now $a+h-\dfrac{\lambda'}{1+\lambda}<a+h-1+\lambda'$ only when $\lambda'>\tau$, values of λ' which do not here come under consideration.

From the cases considered we conclude that $x_{h+1}>0$ and $a+h+\dfrac{1}{x_{h+1}-1}$ $\sim a+h+1$, or to a greater integer. Since $x_{h+1}\geqq 2+\lambda'$, $a+h+\dfrac{1}{x_{h+1}-1}$ $<a+h+1$. Therefore

$$a+h+\frac{1}{x_{h+1}-1}\sim a+h+1. \tag{45}$$

And since

$$a+h+\frac{1}{{}_{h+1}-1}=a+h+1-\frac{x_{h+1}-2}{x_{h+1}-1}, \tag{46}$$

$$\lambda\left(a+\frac{1}{x_1-1}\right)_{\lambda'}=\left(a+h+1,\frac{x_{h+1}-1}{x_{h+1}-2}\right),\quad \frac{1}{1-\lambda'}\leqq x_{h+1}\leqq\frac{1}{\lambda'}+1. \tag{47}$$

The continuation of this λ-development ($\lambda=\lambda'$) comes directly under formula (24) so that it follows at once that

$$\lambda\left(a+\frac{1}{x_1-1}\right)_{\lambda'}=\left(a+h+1,\,2_1,\,2_2,\,\ldots,\,2_{\substack{a-3\\h+1}},\,2+\frac{1}{x_{h+2}-1}\right). \tag{48}$$

It remains only to repeat on $2+\dfrac{1}{x_{h+2}-1}$ the process applied to $a+\dfrac{1}{x_1-1}$ to get a further development of $a+\dfrac{1}{x_1-1}$. The repeated application of this process evidently leads to a $\lambda\left(a+\dfrac{1}{x_1-1}\right)_{\lambda'}$ of one of two types. Either the last element in $\lambda\left(a+\frac{1}{x_1-1}\right)_{\lambda'}$ corresponds to $+2$, in which case the λ-development (48) ultimately coincides with the λ-development (44), or it is true that

$$\frac{1}{1-\lambda'}\leqq x_i\leqq\frac{1}{\lambda'}+1,\qquad i=1,2,\ldots,n,$$

n any positive integer however great, and this latter case the λ-development is of the type (48).

From this result $\lambda\left(-a-\frac{1}{x_1-1}\right)_{\lambda'}$ is at once completely determined so that the λ-developments in the deferred cases under d_1) and d_2) may be at once written down.

The conclusions reached under a), b), c), d), bearing in mind the symmetry of the interval system, completely establishes theorem III.

5.° When $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ and $\lambda(x_0)_{\lambda'}$ are not ultimately alike, then by combining equations (17) and (18) on the one hand and equations (23) and (48) on the other we get the following two forms of $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$, $x_0>0$: viz.

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}=\left(-a-h-1,-2_1,-2_2,\ldots,-2_{\substack{a-3\\h+1}},-2-\frac{1}{x_{h+2}-1}\right),$$
$$a=1,\ x_{h+2}=\frac{1}{1-\lambda'}\cdots\frac{1}{\lambda'}+1;\tag{49}$$

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}=\left(-1,-2_1,-2_2,\ldots,-2_{a_0-2},-a-h-1,-2_1,-2_2,\ldots,\right.$$
$$\left.-2_{\substack{a-3\\h+1}},-2-\frac{1}{x_{h+2}-1}\right),\ a=2,\ x_{h+2}=\frac{1}{1-\lambda'}\cdots\frac{1}{\lambda'}+1.\tag{50}$$

The two forms of $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ represented by equations (49) and (50) may be reduced to one. Let the convention be adopted that when $a_0=1$ and, therefore, $a=1$, then in equation (50) the elements $-1,-2_1,-2_2,\ldots,-2_{a_0-2}$ which precede the element $-a-h-1$ simply do not appear. Under this convention equations (49) and (50) may both be represented by the single equation

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}=\left(-1,-2_1,-2_2,\ldots,-2_{a_0-2},-a-h-1,-2_1,\ldots,\right.$$
$$\left.-2_{\substack{a-3\\h+1}},-2-\frac{1}{x_{h+2}-1}\right),\tag{51}$$

where
$$x_{a+2} = \frac{1}{1-\lambda'} \cdots \frac{1}{\lambda'} + 1,$$

and $a = 1$ or 2 according $a_0 = 1$ or $a_0 > 1$.

When $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ and $\lambda(x_0)_{\lambda'}$, $\lambda_0 > 0$, $0 < \lambda \leqq \tau$, are not ultimately alike,[*] then by the law already applied in equation (51) we have

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = (-1, -2, \ldots, -2_{a_0-2}, -a-h-1, -2_1, \ldots,$$
$$-2_{\substack{a-3\\h+1}}, -3-h_1, -2_1 \ldots, -2_{\substack{a-3\\h+h_1+2}}, -3-h_2 \ldots), \qquad (52)$$

where h is defined by the condition
$$a_1 = a_2 = \ldots = a_h = 2 \ddagger a_{h+1},$$
h_1 by the condition
$$a_{h+2} = a_{h+3} = \ldots = a_{h+h_1+1} = 2 \ddagger a_{h+h_1+2},$$

with similar definitions for $h_2, h_3, \ldots$, and where for $a_0 \geqq 2$, $a = 2$; and for $a_0 = 1$, $a = 1$ and the elements $-1, -2_1, \ldots, -2_{a_0-2}$ immediately preceding the element $-a-h-1$ do not appear in $\lambda(x_0)_{\lambda'}$.

When $x_0 < 0$, then $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ is found by changing the signs of all the elements in equation (52) with the analogous definitions of a and h.

When $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ and $\lambda(x_0)_{\lambda'}$ are not ultimately alike, $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ is called the *associate* of $\lambda(x_0)_{\lambda'}$.

Since the value and order of the ultimate elements in the associate depends only on the value and order of the ultimate elements in the λ-development $(\lambda = \lambda')$, an obvious inference from equation (52), we have the following proposition.

COROLLARY. *If $\lambda(y_0)_{\lambda'}$ and $\lambda(x_0)_{\lambda'}$ are ultimately alike and $\lambda(y_0)_{\lambda'}$ and $\lambda(x_0)_{\lambda'}$ have associated developments, then the associate of $\lambda(y_0)_{\lambda'}$ is ultimately like the associate of $\lambda(x_0)_{\lambda'}$.*

In view of the results now obtained Theorem III may take the form:

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[*] The phrase "ultimately alike" is used here and elsewhere in the sense of the longer phrase "identical from elements of certain ranks on."

THEOREM IV. *For* x_0 *real* $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ *and* $\lambda(x_0)_{\lambda'}$ *are ultimately alike or* $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ *and the associate of* $\lambda(x_0)_{\lambda'}$ *are ultimately alike, and in the latter case* $0<\lambda'\overset{=}{<}\tau$, $\lambda'\overset{=}{<}|x_0|\overset{=}{<}\frac{1}{\lambda'}+1$, $|x_i|\overset{=}{<}\frac{1}{\lambda'}+1$, and sgn. $x_{i+1}=$ sgn. x_i $i=1, 2, \ldots, n-1$.

6.° In the case of the substitution S the following theorem is true:

THEOREM V. *For* x_0 *real* $\lambda(x_0+1)_{\lambda'}$ *and* $\lambda(x_0)_{\lambda'}$ *are ultimately alike, except when for* $0<\lambda'\overset{=}{<}\tau$, *either* $x_0+1\sim 0$ *and* $x_0\sim 0$, *or* $x_0+1\sim 1$ *and* $x_0\sim -1$, *and beginning with* x_1 *all complete quotients in* $\lambda(x_0)_{\lambda'}$ *have the same sign and are in absolute value equal to or less than* $\frac{1}{\lambda'}+1$ *and in this latter case* $\lambda(x_0+1)_{\lambda'}$ *and the associate of* $\lambda(x_0)_{\lambda'}$ *are ultimately alike.*

As in the demonstration of theorem III it is convenient to consider separately the several cases which may arise.

a) $x_0=0\ldots-1$.

In this case it is obvious that

$$x_0+1=a_0+1-\frac{1}{x_1}$$

and that

$$x_0+1\sim a_0+1.$$

Hence

$$\lambda(x_0+1)_{\lambda'}=(a_0+1, a_1, \ldots, a_{n-1}, x_n). \tag{53}$$

b) $x_0=(-1)\ldots-\lambda';\ x_0+1=(0)\ldots(\lambda')$.

By hypothesis

$$x_0=-1-\frac{1}{x_1}$$

and, consequently,

$$x_0+1=-\frac{1}{x_1}\sim 0.$$

Therefore

$$\lambda(x_0+1)_{\lambda'}=(0, a_1, a_2, \ldots, a_{n-1}, x_n). \tag{54}$$

c) $x_0=(-1)\ldots-\lambda',\ x_0+1=\lambda'\ldots 1$.

Again

$$x_0=-1-\frac{1}{x_1}$$

and

$$x_0+1=-\frac{1}{x_1}=1-\frac{x_1+1}{x_1}.$$

Since $x_0 + 1 \sim 1$, then

$$\lambda(x_0 + 1)_{\lambda'} = \left(1,\ 1 - \frac{1}{x_1 - 1}\right). \tag{55}$$

But inasmuch as $x_0 + 1 \sim 1$, then $x_1 < 0$; also, the last equation may be written

$$\lambda(x_0 + 1)_{\lambda'} = \left(1,\ 1 + \frac{1}{-x_1 - 1}\right),$$

where now $-x_1$ corresponds to $+x_1$ in equation (17). Hence applying the conclusion reached in equation (17), we may say that $\lambda(x_0 + 1)_{\lambda'}$ is ultimately like $\lambda(x_0)_{\lambda'}$ or ultimately like the associate of $\lambda(x_0)_{\lambda'}$, a conclusion agreeing with theorem V.

d) $x_0 = (-\lambda') \ldots (0)$, $x_0 + 1 = 0 \ldots (\lambda')$.

Here

$$x_0 = -\frac{1}{x_1}$$

and

$$x_0 + 1 = \frac{x_1 - 1}{x_1} = 0 + \cfrac{1}{1 + \cfrac{1}{x_1 - 1}}.$$

Since

$$x_0 + 1 \sim 0$$

then

$$\lambda(x_0 + 1)_{\lambda'} = \left(0,\ -1 - \frac{1}{x_1 - 1}\right). \tag{56}$$

The continuation of this λ-development ($\lambda = \lambda'$) presents precisely the problem dealt with under equation (17) and hence the conclusions of theorem III apply to equation (56), hence, also, theorem V holds for equation (56).

e) $x_0 = (-\lambda') \ldots 0$, $x_0 + 1 = \lambda' \ldots 1$.

In this case

$$x_0 = -\frac{1}{x_1}$$

and

$$x_0 + 1 = 1 - \frac{1}{x_1}.$$

Hence since

$$x_0 + 1 \sim 1$$

then

$$\lambda(x_0 + 1)_{\lambda'} = (1, a_1, a_2, \ldots, a_{n-1}, x_n). \tag{57}$$

The conclusions reached under a), b), c), d), e), establish theorem V.

7.°. Denote the associate of $\lambda(x_0)_{\lambda'}$ by $A\lambda(x_0)_{\lambda'}$ and the associate of $A\lambda(x_0)_{\lambda'}$ by $AA\lambda(x_0)_{\lambda'}$. For brevity let the symbol $\multimap$ stand for the phrase "is ultimately like." Further read the symbolic expression

$$\lambda(y_0)_{\lambda'} \multimap \begin{matrix} \lambda(x_0)_{\lambda'} \\ A\lambda(x_0)_{\lambda} \end{matrix}$$

disjunctively, thus:— $\lambda(y_0)_{\lambda'}$ is ultimately like $\lambda(x_0)_{\lambda'}$ or ultimately like $A\lambda(x_0)_{\lambda'}$.

Theorem VI. *$AA\lambda(x_0)_{\lambda'}$ is ultimately like $\lambda(x_0)_{\lambda'}$.*

Under certain conditions given in theorem VI.

$$\lambda(T(x_0))_{\lambda'} \multimap A\lambda(x_0)_{\lambda'}.$$

Hence under these conditions since the value and order of the ultimate elements in the associate depends only on the value and order of the ultimate elements in the original λ-development (corollary under theorem III), we have

$$\lambda(TT(x_0))_{\lambda'} \multimap \begin{matrix} A\lambda(x_0)_{\lambda'} \\ AA\lambda(x_0)_{\lambda'} \end{matrix}.$$

But since T^2 is the identical substitution

$$\lambda(TT(x_0))_{\lambda'} \multimap \lambda(x_0)_{\lambda'}.$$

Comparing with the preceding equation

$$AA\lambda(x_0)_{\lambda'} = \lambda(x_0)_{\lambda'}.$$

Let P denote a product of substitutions made up of the factors S and T. Then

Theorem VII. *If $\lambda(P(x_0))_{\lambda'} \multimap \begin{matrix} \lambda(x_0)_{\lambda'} \\ A\lambda(x_0)_{\lambda'} \end{matrix}$, then $\lambda(TP(x_0))_{\lambda'} \multimap \begin{matrix} \lambda(x_0)_{\lambda'} \\ A\lambda(x_0)_{\lambda'} \end{matrix}$.*

For, by theorem IV.

$$\lambda(TP(x_0))_{\lambda'} \multimap \begin{matrix} \lambda(P(x_0))_{\lambda'} \\ A\lambda(P(x_0))_{\lambda'} \end{matrix}.$$

$$\text{a)} \quad \lambda(TP(x_0))_{\lambda'} \multimap \lambda(P(x_0))_{\lambda'}.$$

In this case we have at once by virtue of our hypothesis

$$\lambda(TP(x_0))_{\lambda'} \multimap \begin{matrix} \lambda(x_0)_{\lambda'} \\ A\lambda(x_0)_{\lambda'} \end{matrix}.$$

$$\text{b)} \quad \lambda(TP(x_0))_{\lambda'} \multimap A\lambda(P(x_0))_{\lambda'}.$$

Since the value and order of the ultimate elements in the associate depends only on the value and order of the ultimate elements in the original λ-development (corollary under theorem III),

$$A\lambda(P(x_0))_{\lambda'} \multimap \begin{matrix} A\lambda(x_0)_{\lambda'} \\ AA\lambda(x_0)_{\lambda'} \end{matrix} \multimap \begin{matrix} A\lambda(x_0)_{\lambda'} \\ \lambda(x_0)_{\lambda'} \end{matrix} \text{ by theorem VI.}$$

Hence

$$\lambda\,(TP\,(x))_{\lambda'} \multimap \genfrac{}{}{0pt}{}{\lambda\,(x_0)_{\lambda'}}{A\lambda\,(x_0)_{\lambda'}}.$$

THEOREM VIII. *If* $\lambda\,(P\,(x_0))_{\lambda'} \multimap \genfrac{}{}{0pt}{}{\lambda\,(x_0)_{\lambda'}}{A\lambda\,(x_0)_{\lambda'}}$, *then* $\lambda\,(SP\,(x_0))_{\lambda'} \multimap \genfrac{}{}{0pt}{}{\lambda\,(x_0)_{\lambda'}}{A\lambda\,(x_0)_{\lambda'}}$.

For, by theorem V.

$$\lambda\,(SP\,(x_0))_{\lambda'} \multimap \genfrac{}{}{0pt}{}{\lambda\,(P\,(x_0))_{\lambda'}}{A\lambda\,(P\,(x_0))_{\lambda'}}.$$

$$\text{a)}\quad \lambda\,(SP\,(x_0))_{\lambda'} \multimap \lambda\,(P\,(x_0))_{\lambda'}.$$

By an immediate inference from the hypothesis

$$\lambda\,(SP\,(x_0))_{\lambda'} \multimap \genfrac{}{}{0pt}{}{\lambda\,(x_0)_{\lambda'}}{A\lambda\,(x_0)_{\lambda'}}.$$

$$\text{b)}\quad \lambda\,(SP\,(x_0))_{\lambda'} \multimap A\lambda\,(P\,(x_0))_{\lambda'}.$$

By the corollary under theorem III we have

$$A\lambda\,(P\,(x_0))_{\lambda'} \multimap \genfrac{}{}{0pt}{}{A\lambda\,(x_0)_{\lambda'}}{AA\lambda\,(x_0)_{\lambda'}} \multimap \genfrac{}{}{0pt}{}{A\lambda\,(x_0)_{\lambda'}}{\lambda\,(x_0)_{\lambda'}},\ \text{by theorem VI.}$$

Therefore

$$\lambda\,(SP\,(x_0))_{\lambda'} \multimap \genfrac{}{}{0pt}{}{\lambda\,(x_0)_{\lambda'}}{A\lambda\,(x_0)_{\lambda'}}.$$

Theorems IV, V, VII and VIII provide a basis for the application of the method of complete induction. They justify the following theorem:

THEOREM IX. $$\lambda\,(P\,(x_0))_{\lambda'} \multimap \genfrac{}{}{0pt}{}{\lambda\,(x_0)_{\lambda'}}{A\lambda\,(x_0)_{\lambda'}}.$$

By theorem II every proper unitary substitution is a product of the powers of the elementary substitutions S and T. Hence our results may be summed up as follows:

THEOREM X. *When for x_0 real α, β, γ, δ are integers satisfying the relation*

$$\beta\gamma - \alpha\delta = 1$$

and

$$x'_0 = \frac{\alpha x_0 - \beta}{\gamma x_0 - \delta},$$

then $\lambda\,(x'_0)_{\lambda'}$ and $\lambda\,(x_0)_{\lambda'}$ are ultimately alike or $\lambda\,(x'_0)_{\lambda'}$ and $A\lambda\,(x_0)_{\lambda'}$ are ultimately alike, and in the latter case $0 < \lambda' \leqq \tau$ and from a certain rank on the complete quotients in $\lambda\,(x_0)_{\lambda'}$ all have the same sign and are in absolute value equal to or less than $\frac{1}{\lambda'} + 1$.

8.° Since x_0 and $-x_0$ are improperly equivalent and $\lambda(x_0)_{\lambda'}$ and $\lambda(-x_0)_{\lambda'}$ differ in and only in the signs of their respective elements, the case of improper equivalence may be disposed of at once as in the following theorem:

Theorem XI. *When x'_0 and x_0, two real irrational numbers, are improperly equivalent, then $\lambda(x'_0)_{\lambda'}$ and $\lambda(-x_0)$ are ultimately alike, or $\lambda(x'_0)_{\lambda'}$ and $A\lambda(-x_0)_{\lambda'}$ are ultimately alike, and in the latter case $0 < \lambda' \leqq \tau$ and from a certain rank on the complete quotients in $\lambda(x_0)_{\lambda'}$ all have the same sign and are in absolute value equal to or less than $\frac{1}{\lambda'} + 1$.*

9.° We add here some numerical examples of λ-developments $(\lambda = \lambda')$ having associate λ-developments $\lambda = \lambda'$. The value of x_0 is first given followed by its λ-development $(\lambda = \lambda')$, then a corresponding value x'_0 whose λ-development $(\lambda = \lambda')$ is ultimately like the associate of $\lambda(x_0)_{\lambda'}$ with its λ-development $(\lambda = \lambda')$.

$$\lambda' = \tfrac{1}{2}.$$

$$x_0 = \frac{3+\sqrt{5}}{2}, \quad \lambda(x_0)_{\lambda'} = (3;\ 3;\ 3;\ \ldots),$$

$$x'_0 = -\frac{3+\sqrt{5}}{2}, \quad \lambda(x'_0)_{\lambda'} = (-3;\ -3;\ -3;\ \ldots).$$

$$\lambda' = \tfrac{1}{3}.$$

$$x_0 = \frac{2+\sqrt{2}}{2}, \quad \lambda(x_0)_{\lambda'} = (2,\ 4;\ 2,\ 4;\ \ldots),$$

$$x_0 = -1-\sqrt{2}, \quad \lambda(x'_0)_{\lambda'} = (-3;\ -2,\ -4;\ -2,\ -4;\ \ldots).$$

$$\lambda' = \tfrac{1}{4}.$$

$$x_0 = \frac{5+\sqrt{13}}{6}, \quad \lambda(x_0)_{\lambda'} = (2,\ 2,\ 5;\ 2,\ 2,\ 5;\ \ldots),$$

$$x'_0 = -\frac{3+\sqrt{13}}{2}, \quad \lambda(x'_0)_{\lambda'} = (-4;\ -2,\ -2,\ -5;\ -2,\ -2,\ -5;\ \ldots).$$

$$\lambda' = \tfrac{1}{5}.$$

$$x_0 = \frac{3+\sqrt{5}}{4}, \quad \lambda(x_0)_{\lambda'} = (2,\ 2,\ 2,\ 6;\ 2,\ 2,\ 2,\ 6;\ \ldots),$$

$$x'_0 = -2-\sqrt{5}; \quad \lambda(x'_0)_{\lambda'} = (-5;\ -2,\ -2,\ -2,\ -6;\ -2,\ -2,\ -2,\ -2,\ -6;\ \ldots).$$

$$\lambda' = \tfrac{1}{6}.$$

$$x_0 = \frac{7+\sqrt{29}}{10}, \quad \lambda(x_0)_{\lambda'} = (2,\ 2,\ 2,\ 2,\ 7;\ 2,\ 2,\ 2,\ 2,\ 7;\ \ldots),$$

$$x'_0 = -\frac{5+\sqrt{29}}{7}, \quad \lambda(x'_0)_{\lambda'} = (-6;\ -2,\ -2,\ -2\ -2,\ -7;\ -2,\ -2,\ -2,\ -2,\ -7;\ \ldots).$$

$$\lambda' = \tfrac{1}{7}.$$

$$x_0 = \frac{4+\sqrt{10}}{6}, \quad \lambda(x_0)_{\lambda'} = (2,\ 2,\ 2,\ 2,\ 2,\ 8;\ 2,\ 2,\ 2,\ 2,\ 2,\ 8;\dots),$$

$$x_0' = -3-\sqrt{10}, \quad \lambda(x_0')_{\lambda'} = (-7;-2,-2,-2,-2,-2,-8;$$
$$-2,-2,-2,-2,-2,-8;\dots).$$

$$\lambda' = \tfrac{1}{8}.$$

$$x_0 = \frac{9+\sqrt{53}}{14}, \quad \lambda(x_0)_{\lambda'} = (2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 9;\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 9;\dots),$$

$$x_0' = -\frac{7+\sqrt{53}}{14}, \quad \lambda(x_0')_{\lambda'} = (-8;-2,-2,-2,-2,-2,-2,-9;\dots).$$

$$\lambda' = \tfrac{1}{9}.$$

$$x_0 = \frac{5+\sqrt{17}}{8}, \quad \lambda(x_0)_{\lambda'} = (2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 10;\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 10;\dots),$$

$$x_0' = -4-\sqrt{17}, \quad \lambda(x_0')_{\lambda'} = (-9;-2,-2,-2,-2,-2,-2,-2,-10;\dots).$$

$\lambda' = \dfrac{1}{d}$. d a positive integer greater than 1.

$$x_0 = \frac{d+1+\sqrt{d^2-2d+5}}{2(d-1)}, \quad \lambda(x_0)_{\lambda'} = (2_1,\ 2_2,\dots,\ 2_{d-2},\ d+1;$$
$$2_1,\ 2_2,\dots,\ 2_{d-2},\ d+1;\ \dots),$$

$$x_0' = -\frac{d-1+\sqrt{d^2-2d+5}}{2}, \quad \lambda(x_0')_{\lambda'} = (-d;-2_1,-2_2,\dots,-2_{d-2},-d-1;\dots).$$

In the examples just given

$$x_0' = -\frac{x_0}{x_0-1};$$

and it may be observed that the period in $\lambda(x_0)_{\lambda'}$ is the only one which appears in $A\lambda(x_0)_{\lambda'}$ with no other change than that of the signs of the elements. When d is even, the denominator in the expression for x_0' in terms of d is 2, but when d is odd it reduces to 1.

We add some examples where the periods of $\lambda(x_0')_{\lambda'}$ and of $\lambda(x_0)_{\lambda'}$ differ in other respects than merely a change of the signs of the elements.

$$\lambda' = \tfrac{1}{3}.$$

$$x_0 = 2+\sqrt{3}, \quad \lambda(x_0)_{\lambda'} = (4;\ 4;\ 4;\dots),$$

$$x_0' = -\frac{7+\sqrt{3}}{2}, \quad \lambda(x_0')_{\lambda'} = (-5;-2,-3;-2,-3;\dots).$$

$$\lambda' = \tfrac{1}{4}.$$

$$x_0 = \frac{5+\sqrt{21}}{2};\ \lambda(x_0)_{\lambda'} = (5;\ 5;\ 5;\ \ldots\ldots),$$

$$x_0' = -\frac{27+\sqrt{21}}{6},\ \lambda(x_0')_{\lambda'} = (-6;\ -2,\ -2,\ -3;\ -2,\ -2,\ -3;\ \ldots\ldots).$$

$$\lambda' = \tfrac{1}{5}.$$

$$x_0 = 3+\sqrt{8},\ \lambda(x_0)_{\lambda'} = (6;\ 6;\ 6;\ \ldots\ldots),$$

$$x_0' = -\frac{11+\sqrt{2}}{2},\ \lambda(x_0')_{\lambda'} = (-7;\ -2,\ -2,\ -2,\ -3;\ \ldots\ldots).$$

$$\lambda' = \tfrac{1}{6}.$$

$$x_0 = \frac{7+\sqrt{45}}{2},\ \lambda(x_0)_{\lambda'} = (7;\ 7;\ 7;\ \ldots\ldots),$$

$$x_0' = -\frac{65+\sqrt{45}}{10},\ \lambda(x_0')_{\lambda'} = (-8;\ -2,\ -2,\ -2,\ -2,\ -3;\ \ldots\ldots).$$

$$\lambda' = \frac{1}{d},\ d \text{ a positive integer greater than one.}$$

$$x_0 = \frac{d+1+\sqrt{d^2+2d-3}}{2},\ \lambda(x_0)_{\lambda'} = (d+1;\ d+1;\ d+1;\ \ldots\ldots),$$

$$x_0' = -\frac{(d+1)(2d+1)+\sqrt{d^2+2d-3}}{2(d-1)},\ \lambda(x_0')_{\lambda'} = (-d-2;\ -2_1,\ -2_2,\ \ldots\ldots,\ d_{d-2},\ -3;\ \ldots\ldots).$$

In these examples

$$x_0' = \frac{dx_0 - d^2 + 1}{-x_0 + d}.$$

Again consider these examples:

$$\lambda' = \tfrac{1}{3}.$$

$$x_0 = \frac{6+\sqrt{24}}{3},\ \lambda(x_0)_{\lambda'} = (4,\ 3;\ 4,\ 3;\ 4,\ 3;\ \ldots\ldots),$$

$$x_0' = -\frac{18+\sqrt{24}}{5};\ \lambda(x_0')_{\lambda'} = (-5;\ -3,\ -2,\ -3;\ -3,\ -2,\ -3;\ \ldots\ldots).$$

$$\lambda' = \tfrac{1}{4}.$$

$$x_0 = \frac{5+\sqrt{20}}{2},\ \lambda(x_0)_{\lambda'} = (5,\ 4;\ 5,\ 4;\ 5,\ 4;\ \ldots\ldots),$$

$$x_0' = -\frac{50+\sqrt{80}}{11},\ \lambda(x_0')_{\lambda'} = (-6;\ -2,\ -3,\ -2,\ -2,\ -3;\ \ldots\ldots).$$

$$\lambda = \tfrac{1}{5}.$$

$$x_0 = \frac{15 + \sqrt{195}}{5},\ \lambda(x_0)_{\lambda'} = (6,\ 5;\ 6,\ 5;\ 6,\ 5;\ \ldots),$$

$$x'_0 = -\frac{105 + \sqrt{195}}{19},\ \lambda(x'_0)_{\lambda'} = (-7,\ -2,\ -2,\ -3,\ -2,\ -2,\ -2,\ -3;$$

$$\lambda = \frac{1}{d}\ d \text{ a positive integer greater than one.}$$

$$x_0 = \frac{d(d+1) + \sqrt{(d^2+d)(d^2+d-4)}}{2d},\ \lambda(x_0)_{\lambda'} = (d+1,\ d;\ d+1,\ d;\ d+1,\ d;\ \ldots),$$

$$x'_0 = \frac{d(2d^2-d-3) + \sqrt{(d^2+d)(d^2+d-4)}}{2(d^2-d+1)};\ \lambda(x'_0)_{\lambda'} = (-d-2;\ -2_1,\ -2_2,\ \ldots,\ -2_{d-3},\ -3,\ -2_1,\ \ldots,\ -2_{d-2},\ -3;\ \ldots).$$

In this list as in the preceding

$$x'_0 = \frac{dx_0 - d^2 + 1}{-x_0 + d}.$$

In H_I and H_{II} and in the preceding lists the λ-developments $(\lambda = \lambda')$ considered are all periodic.

Let us now consider an example of a non-periodic λ-development $(\lambda = \lambda')$ having an associated λ-development $(\lambda = \lambda')$.

Denote by x_0 the infinite λ-sequence derived from the continued fraction

$$x_{n(3n+1)} = (3_1,\ 4_1,\ 4_2,\ 3_1,\ 3_2,\ 3_3;\ 4_1,\ 4_2,\ 4_3,\ 4_4,\ \ldots,\ 3_1,\ 3_2,\ \ldots,\ 3_{2n-1},\ 4_1,\ 4_2,\ \ldots,\ 4_{2n});$$

which is a λ-development for $\lambda' = \frac{1}{4}$, by taking the limits of both members of this equation as $n = \infty$. Then

$$\lambda(x_0)_{\lambda'=\frac{1}{4}} = (3,\ 4_1,\ 4_2,\ 3_1,\ 3_2,\ 3_3,\ 4_1,\ 4_2,\ 4_3,\ 4_4,\ 3_1,\ 3_2,\ 3_3,\ 3_4,\ 3_5,\ 4_1,\ 4_2,\ 4_3,\ 4_4,\ 4_5,\ 4_6,\ \ldots).$$

Now let

$$x_0 = -\frac{1}{x_0},$$

then

$$\lambda(x'_0)_{\lambda'=\frac{1}{4}} = (-1,\ -2,\ -3,\ -2,\ -3,\ -2;\ -3,\ -3,\ -3;\ -3,\ -2,\ -3,\ -2,\ -3,\ -2,\ -3,\ -2;\ -3,\ -3,\ -3,\ -3,\ -3;\ -3,\ -2,\ -3,\ -2,\ -3,\ -2,\ -3,\ -2,\ -3,\ -2,\ -3,\ -2,\ \ldots).$$

It may be remarked that a set of 3's in $\lambda(x_0)_{\lambda'}$ goes over into a set of $-$ 3's in $A\lambda(x_0)_{\lambda'}$ i. e. in $\lambda(x'_0)_{\lambda'}$; that every 4 in $\lambda(x'_0)_{\lambda'}$ gives rise to a sequence $-3,\ -2$ in $A\lambda(x_0)_{\lambda'}$. The semicolons, departing from customary usage, serve to separate in the associate the sequences arising from sets of 3's in $\lambda(x_0)_{\lambda'}$ from the sequences arising from sets of 4's.

In the manner indicated any number of non-periodic λ-developments $(\lambda=\lambda')$ having associates may be set up. The property of having associates is not, therefore, characteristic of quadratic irrationalities.

Let it be observed that this process of associated λ-developments $(\lambda=\lambda')$ appplies as well to terminating as to infinite λ-developments $(\lambda=\lambda')$.

For example let

$$\lambda(x_0)_{\lambda'=4}=(3,\ 4_1,\ 4_2,\ 3_1,\ 3_2,\ 3_3,\ 4_1,\ 4_2,\ 4_3,\ 4_4,\ \ldots,\ 3_1,\ 3_2,\ \ldots,\\ 3_{2n-1},\ 4_1,\ 4_2,\ \ldots,\ 4_{2n}),$$

and let

$$x_0'=-\frac{1}{x_0}.$$

Then

$$\lambda(x_0')_{\lambda'=4}=(-1,\ -2;\ -3_1,\ -3_2,\ -3_3,\ -3,\ -2,\ -3,\ -2,\ -3,\ -2,\ -3,\ -2,\\ \ldots,\ -3_1,\ -3_2,\ \ldots,\ -3_{2n-1},\ -3_1,\ -2_1,\ -3_2,\ -2_2,\ \ldots,\ -3_{2n},\ -2_{2n},\ -2_{2n}),$$

where the subscript to an element in $\lambda(x_0')_{\lambda'}$ indicates that the element is derived from the element with the same subscript in $\lambda(x_0)_{\lambda'}$.

Twisted Curves whose Tangents belong to a Linear Complex.

By Virgil Snyder.

It was shown by Lie that the problem of finding the curves whose tangents belong to a non-special linear complex is co-extensive with that of the enumeration of minimum developables. By duality, the latter reduces to the problem of curves on a quadric cone, hence if these latter curves be known, the corresponding complex curves are determined by means of two transformations.

Since the properties of lying on a quadric cone and of belonging to a linear complex are purely projective, it will be desirable to replace the minimum developable by any developable on which a conic lies, and to avoid both metrical and imaginary elements of the transformation. It is the purpose of this paper to analyze the singularities of the Noether line-point transformation and of duality, then to give a classification of complex curves of orders one to six, and to discuss the forms of singularities which complex curves can have.

1. *Curves on a quadric cone.*

Let an algebraic curves pass b times through the vertex of a quadric cone K_2 and cut each generator in a other points. Its order is then $2a+b$. By projecting this curve from an arbitrary point O of K_2 through which it does not pass, and cutting the projecting cone with an arbitrary plane π we obtain a plane curve of order $2a+b$ in $(1, 1)$ correspondence with the curve on K_2. Let the generator through O cut π in O_1. It cuts the curve on K_2 in $a+b$ points, the tangents to a of which lie in the tangent plane to K_2 along OO_1. In the plane curve we have a branches touching each other, with an independent b-fold point at the point of tangency. This singularity is equivalent to

$$a(a-1)+\frac{b}{2}(b-1)+ab$$

double points. The plane curve can have no other singularities except the projections of actual nodes and cusps of the curve on K_2. The class of the plane curve is equal to the rank of the twisted curve, hence

$$r = 2a(a+b) - 2H - 3\beta$$
$$p = (a-1)(a+b-1) - H - \beta.^*$$

An arbitrary line will cut r tangents to c_{2a+b}. If the line pass through the vertex of K_2, $2a$ of these tangents will evidently lie in the tangent planes to K_2 through the line. The b tangents at the vertex will each count for two, hence there are $2a(a+b) - 2H - 3\beta - 2(a+b)$ generators of K_2 which touch the curve, the point of tangency not being at the vertex. The osculating planes at these points are stationary planes passing through the vertex. The other stationary planes do not pass through the vertex. The osculating plane at a simple cusp must pass through the vertex, as must also the osculating plane at at a point of inflexion, the latter being a plane of six point contact, being the equivalent of three stationary planes. The inflexional tangent must pass through the vertex. A cusp or linear inflexion may exist at the vertex; higher point singularities may exist at other points of the curve, without requiring the osculating plane to pass through the vertex. The effect of these singularities will be the same as that of their projections in π.

If the equation of K_2 be $y^2 - 4xz = 0$, and the coordinates of O be $(0, 1, 0, 0)$ the relation between the point (x, y, z, w) on the cone and the point (ξ, η, ζ) in π may be represented thus:

$$\rho w = \xi\zeta, \quad \rho x = \xi^2, \quad \rho y = 2\xi\eta, \quad \rho z = \eta^2.$$

The equation of the tacnodal tangent, intersection of π and the tangent plane to K_2 along OO_1 is $\xi = 0$, the point of tangency being $(0, 0, 1)$. A plane curve of order $2a + b$ having a branches touching each other and another b-fold point at the point of contact is the projection of a conical c_{2a+b}. Every other singularity of the plane curve will project into a similar singularity of the conical curve.

2. *Duality.*

By means of the point (x, y, z, w) and plane (u, v, s, t) duality wherein $ux + vy + sz + tw = 0$, the cone $y^2 - 4xz = 0$ goes over into the conic c_2:

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[*] The letters r, p, H, β and others that will be used have the same meaning as in Pascal's (German) Repertorium, vol. 2, p. 228, col. 4. This formula was proved by Ed. Weyr and by Baule by means of abelian integrals. It was generalized to apply to curves on a cone of any order by Sturm, *Math. Annalen*, vol. 19, p. 487, where the preceding memoirs are also cited.

$y^2 - xz = 0$, $w = 0$, the vertex $(0, 0, 0, 1)$ into the plane $w = 0$, the generators of K_2 into the tangents of c_2. The curve c_{2a+b} becomes a developable D_r of rank r, having c_2 for a-fold curve, having b-tangents with w for torsal plane, and $2a(a+b) - 2H - 3\beta - 2(a+b)$ other tangents in $w = 0$ which are simple generators of the developable. Stationary planes go into cusps; those passing through the vertex into those on c_2; cusps go into stationary planes, the point of contact being on c_2. The various singularities of c_{2a+b} which have no special relation to the vertex of K_2 go over into dual singularities having no particular relation to $w = 0$.

3. *The Noether line-point transformation.*

The details of this transformation were given in my Göttingen dissertation,* but in metrical form, and with reference to spheres rather than minimum lines as generators.

Consider the two bi-linear equations

$$x_1 x + y_1 y + w_1 w = 0,$$
$$y_1 x + z_1 y + w_1 z = 0,$$

in which x, y, z, w and x_1, y_1, z_1, w_1 are coordinates of points in two spaces S, S_1. When (x_1) describes S_1 the corresponding line in S describes the linear complex $\psi \equiv p_{14} + p_{23} = 0$. When (x) describes S, the corresponding line in S_1 describes the quadratic complex $\Omega \equiv p_{14} p_{34} - p_{42}^2 = 0$, consisting of the lines which cut the conic $y_1^2 - x_1 z_1 = 0$, $w_1 = 0$.†

The lines of a pencil in ψ go into the points of the line in Ω which is the image of the vertex. A curve in ψ goes over into a developable in Ω. The ∞^2 lines of a linear congruence contained in ψ, defined by

$$a p_{12} + b p_{34} + c p_{31} + d p_{24} + e p_{14} + f p_{23} = 0$$

become the ∞^2 points of the quadric surface R_2,

$$b(x_1 z_1 - y_1^2) - w_1 (a w_1 + c z_1 + d x_1 + (f - e) y_1) = 0,$$

which may be said to be the image of the two directrices of the congruence. These directrices are conjugate polars as to ψ, hence to any line in S corresponds an R_2 in S_1 containing c_2; the second system of generators of R_2 are the images of the points of the conjugate polar of the given line as to ψ. When the directrices coincide, the single directrix belongs to ψ; its points go over into the quadric cone whose vertex is the image of the given line, and which contains c_2. If the directrices of the congruence cut l, $x=0$, $y=0$, i.e., if $b=0$, R_2 breaks up into $w_1=0$ and the plane $a w_1 + c z_1 + d x_1 + (f-e) y_1 = 0$; the points of the directrices go over into the two pencils in the second plane, whose vertices are at the points in which the plane cuts c_2. A directrix belonging to ψ and cutting l goes over into a tangent plane to c_2.

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* Ueber die linearen Complexe der Lie'schen Kugelgeometrie, Göttingen, 1895. Many of the details are also given in Lie-Scheffers, Berührungstransformationen, 1897.

† This form of the equations is given by Wiman, Klassifikation af regelytorna af sjette Graden, Lund dissertation, 1892. The transformation of singularities is not discussed. See also Jessop's Treatise on the line complex, pp. 121–8.

The ∞^2 lines of ψ which cut l go over into the ∞^2 points of $w_1=0$. Conversely, all the points of $w_1=0$ go over into the same line l. The relation between the points in S_1 and directions from points of l is expressed by $x_1 : y_1 : z_1 = d p_{24} : d p_{23} : d p_{31}$. To a point x_1, y_1 on c_2 correspond lines of ψ in $x_1 x + y_1 y = 0$, the vertex being on l. To the ∞^2 lines in S_1 through this point correspond the ∞^2 points of the plane $x_1 x + y_1 y = 0$. To the lines of a pencil containing the tangent to c_2 and vertex on c_2 correspond ∞^1 points on a line of ψ in the plane. When a curve in S_1 has a double point on c_2, it goes over into two lines of ψ from the same point of l. When the plane of the two tangents contains the tangent to c_2 at the double point the image lines in S coincide.

A line in $w_1=0$ cuts c_2 in two points, but we need not consider such cases here, since every generator of a developable containing c_2 and lying in the plane of the latter must be tangent to c_2. If then the developable be of order r and contains c_2 as an a-fold line, the corresponding curve in S will be of order $r-a$, having $r-2a$ points on l.

Cusps and linear inflexions in S_1 go into linear inflexions and cusps in S, provided they do not lie on c_2. A cusp of D_r on c_2 goes into a point of c_{r-a} on l, and a point of stationary contact on c_2 goes over into a line of ψ cutting l which has a point of tangency not on l. A generator of D_r in w_1, and having w_1 for torsal plane goes into a point of c_{r-a} on l, having l for tangent. A double tangent plane goes into a line of ψ cutting l which is a bisecant of c_{r-a}.

On combining these two transformations we see that every curve on K_2 goes over into a curve in ψ and conversely. In this depiction, any complex whatever that is not special can by linear point transformation be reduced to this form ψ. The same complex curve may give rise to different curves on K_2 by choosing l differently. Any line of the complex to which a curve may belong can be taken as l, and any conic in S_1 may be taken for c_2.

A curve of order $2a+b$ on K_2 goes over into a complex curve of order $2a(a+b)-2H-3\beta-a$. The line l has $2a(a+b)-2H-3\beta-2a$ points upon the curve, of which $2b$ are absorbed in b points of contact, images of the b tangents to c_{2a+b} at the vertex of K_2. The other points of the complex curve on l are images of the stationary planes which pass through the vertex of K_2. There are β tangents of the complex curve which cut l, and also H bisecants belonging to ψ cut l. The genus of the complex curve is the same as that of the conical curve. To every stationary plane of the conical curve not passing through the vertex corresponds an inflexion of the complex curve, hence we have the following theorem: *At the point of contact of a stationary plane of a curve contained in a linear complex the tangent line has at least three point contact.*[*]

If a tangent having more than two-point contact with the conical curve does not pass through the vertex, the point of contact must be a singular point, hence: *The osculating plane at a cusp of a curve contained in a linear complex must have at least five-point contact with the curve at the cusp.*

Finally, since every complex curve is self-dual in such a way that every generator of the developable formed by its tangents goes into itself when inverted in the complex[†] it follows that $m=n$, $\alpha=\beta$, $H=S=0$, $h=g$, $x=y$. Further, the cusps and points of stationary contact must be in united position, provided the latter is not an inflexion.[‡]

4. *Particular examples.*

The c_3, $a=1$, $b=1$ or $(1, 1)$ on K_2 goes over into c_3 in ψ, l being a tangent.

A nodal c_4 $(2, 0)$, vertex not at the node, is of rank 6; two tangents pass through the vertex and it has two other stationary planes. It becomes a c_4 having two linear inflexions; l is a bisecant of c_4 and lies in the osculating plane of each point of intersection, hence no other tangents can cut l.

These two curves were long since known to belong to a complex. Now consider the c_4 having its node at the vertex of K_2, type $(1, 2)$. No tangents pass through the vertex except the two at the vertex. We have in ψ a c_5 of which l is a bitangent and having four inflexions.*

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[*] This theorem can also be proved geometrically. Another proof was given by Picard, Application de la théorie des complexes linéaires à l'étude des surfaces et des courbes gauches. *Annales de l'école normale*, 1877.

[†] See Wilczynski, *Math. Annalen*, vol. 58, p. 249, and Sisam, *Bulletin Amer. Math. Society*, vol. 10, p. 440.

[‡] The word duality is used in a much broader sense by Fiedler in the Vierteljahresschrift der Naturforsch. Gesellschaft in Zürich, vol. 20, in which he enumerates possible self-dual curves, simply according to the Cayley characteristics. Complex curves are not discussed.

The cuspidal c_4, vertex not at cusp goes over into c_3 having one point on l; one tangent to c_3 cuts l. If the cusp be at the vertex, the complex curve is a c_4 having l for one inflexional tangent, and one other.

An interesting curve is afforded by the quartic of the first kind (2, 2). The complex curve is of order six and genus one. The line l is a quadri-secant, the points of intersection being harmonic. The 12 inflexional tangents are arranged in three tetrads such that the two transversals of a tetrad also cut l and are conjugate polars as to ψ. The lines in the stationary planes which cut l and belong to ψ are so arranged that a line of ψ cutting any three will also cut a fourth. The conical curve c_{2n+1}:

$$x=(2n+1)\lambda^{2n+1}, \quad y=2(2n+1)\lambda^{n+1}, \quad z=(2n+1)\lambda, \quad w=2n,$$

has all points of the plane $x=0$ at the vertex of the cone, the generator $x=0$, $y=0$ being a tangent of $(n+1)$-point contact. The point (1, 0, 0, 0) is an $(n-1)$-fold point, the osculating plane $w=0$ having $(2n+1)$-point contact, the tangent line having $2n$-point contact. It goes into the complex curve:

$$x=(2n+1)\lambda^{2n+1}, \quad y=(2n+1)\lambda^{n+1}, \quad z=(2n+1)\lambda^{n}, \quad w=-1,$$

which has two n-fold points, the tangents having $(n+1)$-point contact, and osculating planes having $(2n+1)$-point contact. The points are (0, 0, 0, 1) and (1, 0, 0, 0). Thus a cusp of this nature on the complex curve and having the fundamental line for tangent goes into the vertex of the cone K_2, the tangent having $(n+1)$-point contact, and the point itself simple on the conical curve. The same singularity, but having no particular relation to the fundamental line goes into a singular point of the same order as that on the complex curve, but the tangent has maximum contact. Neither the tangent nor the osculating plane passes through the vertex of the cone. Similarly, the conical curve

$$x=n\lambda^{n}, \quad y=2n\lambda^{n-1}, \quad z=n\lambda^{n-2}, \quad w=2,$$

having an $(n-2)$-fold point at the vertex, and a stationary plane of n-point contact at (0, 0, 0, 1), the tangent having simple contact, goes into the complex curve

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* The rational quintic with four inflexions and a double tangent has also been found by Dr. Colpitts in his Enumeration of twisted quintic curves, Cornell dissertation, 1906, by a totally different method.

$$x = n\lambda^n, \quad y = n\lambda^{n-1}, \quad z = n\lambda, \quad w = 2 - n,$$

having two tangents of $(n-1)$-point contact, but no multiple points.*

As an example of a less symmetric curve, the conical c_6: $x = \lambda^6$, $y = 20\lambda^3$; $z = 100$, $w = 6\lambda$ goes into the complex c_6: $x = 2\lambda^5$, $y = 20\lambda^2$, $z = \lambda^3$; $w = 2$.

It is thus seen that the tangents to the conical curve go into the points of the complex curve, and the osculating planes of the conical curve go into the tangents of the complex curve. The vertex goes into l. The points of c_{2a+b} go into lines of ψ cutting l, determined by the osculating plane. These lines form a ruled surface contained in a special linear congruence having l for single directrix, and the complex curve for asymptotic line. The order of the surface is $2a + b$; l is an a-fold directrix and an a-fold generator; b generators lie in any plane through l. The order of the complex curve is (generally) equal to the class of a plane section of this surface.

5. *Asymptotic lines on certain ruled surfaces.*

The asymptotic lines of an algebraic ruled surface contained in a linear congruence are algebraic, belong to a complex containing the congruence, cut every generator twice and are of order equal to the class of a plane section of the surface. We can therefore determine their genus by Segre's theorem.† If the surface have an m-fold directrix and a distinct n-fold directrix, δ double, and $\varkappa$ cuspidal generators, the genus p is defined by

$$p = 4mn - 3(m + n) + 1 - 4\delta - 5\varkappa.$$

The pinch-points are inflexions for all the asymptotic lines, the common inflexional tangent being the torsal generators. All the generators of the surface are principal chords, i. e., the osculating plane at each point of intersection passes through the other point.

If these curves be projected from any point on the m-fold directrix, the plane projection will have a $2(m^2 - m - \delta - \varkappa)$-fold point, each branch having an inflexion at the multiple point and n other nodes lying on a line, the nodal tangents being inflexional. All the remaining points in which the line joining the nodes cuts the curves are points of inflexion. The various asymptotic lines of the surface will have the inflexional tangents at the multiple point in common, as well as the simple inflexions and the nodes, but the inflexional tangents at the nodes will form a quadratic involution.

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* These two complex curves have other remarkable properties which were discussed in the *Journal*, vol. 28.

† *Math. Ann.* vol. 34, p. 3 (1889).

Conversely, given any complex curve, we can construct a ruled surface having the given curve for asymptotic line. Let h be a line not belonging to the complex. Connect each point P of c with the point Q on l in which l pierces the osculating plane of P. The line PQ will generate a surface having h, h' for directrices, h' being the conjugate polar of h as to the complex. By choosing h in various ways, the given curve can be made of different order than that of the other asymptotic lines. In particular, the line can be so chosen that all the asymptotic lines are composite.

6. *Maximum genus of complex curves.*

The question of the maximum genus of a complex curve of given order is equivalent to that of the maximum number of cusps which a conical curve of given order can have. The latter in turn is replaced by the problem of plane curves: given a curve of order $2a$, having a branches touching each other at a given point, required the maximum number of cusps which the curve may have, since it is easy to see that we need only consider curves of complete intersection, which do not pass through the vertex of the cone. This problem has not been solved, it is simply a special case of the classic problem of the number of constants which the double points of a curve absorb, but an upper limit can be fixed in each case. Thus, if $a = 4$, $\beta \not> 7$; if $\beta = 7$, it can have no double points. This gives rise to a complex curve of order 7 and genus 2. If $a = 6$, $\beta = 18$, $H = 2$ exists and gives rise to a complex curve of order 8 and genus 5. It may be obtained as the intersection of K_2 and a cone having for base a curve of order 6 and class 3, two tangents lying in tangent planes of K_2. The asymptotic lines of the quartic scroll having two distinct double directrices are of this type. They have forty inflexional tangents, eight of which are generators of the scroll. In general, l is a bisecant and is intersected by 18 tangents, making its rank 24.

By reducing to simple curves of maximum genus we obtain the following upper limits, but for larger values of a the actual upper limit is probably lower:

a	2	3	4	5	6	7	8	9	10	11	12
β	1	4	7	12	18	26	36	47	59	72	88

The corresponding complex curves become:

order	3	4	5	6	7	8	9	10
max. genus	0	0	0	1	2	5	10	15

7. *Complex curves of orders three to six.*

Every twisted cubic belongs to a linear complex. The quartic with two inflexions is the only form, although the cuspidal quartic is selfdual in the more general sense. The complex quintics are unicursal; they have four inflexions, with or without a double tangent, a four-point tangent with two inflexions or another four-point tangent, a cusp with osculating plane having five-point contact and two inflexions, or another cusp. The forms having four inflexions, or one cusp can not lie on a quadric.

The conical curves which transform into rational sextics contained in ψ are of the forms:

(2, 1), $H = 2$, this goes into $v = 6$, l is $t + s_2$.
(2, 2), $\beta = 2$, $H = 1$, l is t_2.
(1, 4), all tangents coincident, l is tangent of 5-point contact.
(3, 0), $\beta = 1$, $H = 3$, l is s_3.
(3, 1), $\beta = 3$, $H = 3$, l is $t + s_1$.
(2, 3), $\beta = 2$, $H = 0$ (coincident tangents), l has 4-point coincident contact.
(1, 3), two coincident tangents, l is simple and distinct inflexional tangent.*

Of these forms, all but the first and last may also have a double osculating plane, giving rise to a bitangent of the complex curve. Stationary planes having 5-point contact can occur in all, and having 6-point contact in all but the first and last. Combining these with possible point singularities, we obtain the following forms:

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* The statement made by Van der Vries, *Proceedings Amer. Acad.* vol. 38 (1903), p. 524, that no two of the tangents at a triple-point on a twisted quintic can coincide is incorrect. The equations of our curve may be written $x : y : z : w = \lambda^4(\lambda - 1) : 2\lambda^3(\lambda - 2)^2 : \lambda(\lambda - 1)^3 : 1$ in which case the complex curve has another tangent of four-point contact.

$\alpha = \beta = 0$.
$v = 6$, $\omega = 0$ or 1.
$v = 4$, one 4-point contact tangent, $\omega = 0$ or 1, or one v is tangent.
$v = 2$, two 4-point contact tangents.
$v = 0$, three 4-point contact tangents.
$v = 3$, one 5-point contact tangent.
$v = 1$, one 4-, one 5-point contact tangent.
$v = 0$, two 5-point contact tangents.

$\alpha = \beta = 1$.
$v = 4$, $\omega = 0$ or 1.
$v = 2$, one 4-point.
$v = 0$, two 4-point.

$\alpha = \beta = 2$. $v = 2$; $v = 0$, one 4-point.
$\alpha = \beta = 3$. $v = 0$.

The sextics of genus 1 have already been discussed.

CORNELL UNIVERSITY, *May* 9, 1906.

Groups in which every Subgroup is either Abelian or Dihedral.

By G. A. Miller.

As the groups in which every subgroup is abelian are known[*] we shall assume that every group G under consideration contains at least one non-abelian dihedral subgroup. Our first object is to prove that G is always solvable. Since every subgroup and every quotient group of G is either abelian, or has the properties which have been assumed with respect to G, it follows that G is either solvable or there is some G which is simple. It is therefore only necessary to show that G cannot be a simple group. We shall do this by assuming that G is simple and showing that this assumption necessarily leads to a contradiction. Until G is proved solvable it will be assumed that it is simple.

Suppose that G is represented as a substitution group in such a way that a Sylow subgroup of odd order (P) is composed of all the substitutions of G which do not involve a given letter. The subgroup P is abelian. If it were maximal G would be of degree n and class $n-1$, since P would be of degree $n-1$ and could have only the identity in common with any of its conjugates. As a group of degree n and class $n-1$ contains an invariant subgroup, P cannot be maximal and hence it must be contained in a dihedral subgroup of G. The latter statement follows from the fact that no abelian subgroup can be maximal in a simple group. Hence we have proved the lemma that *every Sylow subgroup of odd order contained in G is cyclic and every abelian subgroup of G is contained in some dihedral subgroup whenever G is simple.*

Since the Sylow subgroup of order 2^m is contained in a dihedral group it must itself be dihedral and hence involves operators of order 2^{m-1}. Suppose now that G is represented as a primitive substitution group in such a way that the subgroup (G_1) which is composed of all its substitutions which do not involve a given letter involves a Sylow subgroup of order 2^m. There is one invariant operator (s_1) of order 2 in G_1* and all its operators whose order exceeds 2 are of degree $n-1$, n being the degree of G. We proceed to prove that the degree of s_1 is less than $n-1$.

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[*] Transactions of the American Mathematical Society, vol. 4 (1903), p. 398.

Since G is simple and G_1 is a maximal subgroup, s_1 is not commutative with any operators of G, except those of G_1. Hence s_1 has n conjugates under G. These n conjugates include all the operators of G which are powers of operators of order 2^α, $\alpha > 1$, since all the cyclic subgroups of order 2^α are conjugate under G. If G contained any operators of order 2 besides those which are conjugate with s_1 it would contain negative substitutions if it were represented as a transitive substitution group in such a way that one of these operators of order 2 would generate the subgroup involving all the substitutions which do not contain a given letter. That is, G contains a single set of conjugates of order 2 and the number of these conjugates is n.

From the preceding paragraph it follows that G would be of class $n-1$ if s_1 were of degree $n-1$. As a simple group cannot be of degree n and of class $n-1$ it follows that the degree of s_1 is less than $n-1$. The number of four-groups in G_1 is $g_1/4$, g_1 being the order of G_1. As every operator of order 2 in G is contained in exactly $g_1/4$ four-groups, the total number of these subgroups is $ng_1/12 = g/12$, g being the order of G. If they would all be conjugate each of them would be invariant under a group of order 12. As such a subgroup is neither dihedral nor contained in a dihedral group it follows that G cannot be simple unless its order is divisible by 8.

If g were divisible by 8 G would contain two operators of order 4, (t_1, t_2) such that the square of each would transform the other into its inverse. That is, we would have

$$t_1^4 = t_2^4 = 1, \qquad t_1^2 t_2 t_1^2 = t_2^3, \qquad t_2^2 t_1 t_2^2 = t_1^3, \qquad t_1^2 \neq 1.$$

From these relations it follows that $\{t_1^2, t_2^2\}$ is the four-group. Moreover: the equations

$$(t_1 t_2)^{-1} t_1^2 t_1 t_2 = t_1^2 t_2^2, \qquad (t_1 t_2)^{-1} t_1^2 t_2^2 t_1 t_2 = t_2^2,$$

show that the subgroup generated by $s_1 s_2$, s_1^2, s_2^2 is neither abelian nor dihedral. This proves that G is composite. In other words, *If every non-abelian sub-group of a group is dihedral the group is always solvable.*

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* The order of G_1 is divisible by 4 since the order of G cannot be twice an odd number.

If the order of a dihedral group is divisible by 4, its non-invariant operators of order 2 may be replaced by operators of order 4 which transform the operators of the group in the same way as the operators of order 2 did. That is, the group of cogredient isomorphisms remains unchanged. The octic and the quaternion groups are simple illustrations of such groups. If we call the latter type associate-dihedral, the arguments used above prove also that *if every non-abelian sub-group of a group is either dihedral or associate-dihedral the group is solvable.*

Since G is solvable it contains an invariant subgroup of prime index (p). If $p > 2$ the corresponding invariant subgroup (H) involves all the operators of G whose order is of the form 2^α. There is some operator (s) of order p^β which is found in G but not in H. If H were abelian it would contain some operators of order 2 which would transform an operator of order p into its inverse since G contains at least one non-abelian dihedral subgroup. This is impossible since every two operators of order 2 in H are supposed to be commutative. Hence H must be dihedral whenever $p > 2$. As s is commutative with every operator in the cyclic subgroup of half the order of H it follows that *G is either the direct product of the octic group and a cyclic group of order p, or it is the direct product of a cyclic group of order p and a dihedral group of order $2q$; q being an odd prime, whenever $p > 2$.*

When $p = 2$ G may be a dihedral group. As the properties of such a G are well known we shall assume for the present that G is not dihedral but that H is abelian. Some operators of order 2 which are contained in the non-abelian dihedral subgroup (D) cannot occur in H since H is abelian. We may therefore assume that S is of order 2. The considerations of this case may be simplified by means of the following theorem, which is of interest on account of its great generality. *If an operator of order 2 transforms a group of order 2^m into itself and is commutative with only two of its operators the group of order 2^m is cyclic, dihedral, or associate dihedral.*

The proof of this theorem follows almost directly from the facts that the given group contains (whenever $m > 3$) an abelian non-cyclic invariant subgroup of order 8 whenever it contains more than one cyclic subgroup of order 8,[*] and an operator of order 2 which transforms an abelian non-cyclic group of order 8 into itself is commutative with at least four of its operators. The group of order 2^m, $m > 3$, which contains only one cyclic group of order 8 and is not mentioned in the theorem can clearly not be transformed into itself by an operator of order 2 which is commutative with only two of its operators. Hence the proof of the theorem is complete since the four-group is dihedral and the octic and quaternion groups are respectively dihedral and associate dihedral.

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[*] Transactions of the American Mathematical Society, vol. 6 (1905), p. 60.

If G would contain an invariant operator of odd prime order it would also involve an invariant subgroup of odd prime index since the given invariant operator could not be contained in D. As this case has been considered we may assume that s transforms into its inverse every operator of odd prime order contained in H. The truth of the last statement follows directly from the theorem: If an operator (t) transforms commutative operators among themselves it is commutative with the continued product of any complete set of conjugates under t. In particular, *if an operator of order two transforms any operator into one which is commutative with it, the operator of order two is itself commutative with the product of this operator and its conjugate.*

Suppose that G contains more than two invariant operators. From the preceding paragraph it follows that it is only necessary to consider the case when there is an invariant operator (i) of order 2^β in G and i does not occur in D. Since the direct product of the group of order 2 and a dihedral group whose order is not divisible by 4 is dihedral, we may assume that the order of D is divisible by 4. If i is of order 2 all the subgroups of D whose orders are divisible by 4 must be abelian. Hence the order of D is not divisible by 8 unless D is the octic group. As the order of D is supposed to be divisible by 4 it can contain only one subgroup of odd order. That is, *if G contains at least two invariant operators of order 2 it is the direct product of a group of order 2 and either the octic group or the dihedral group of order $4p$, p being an odd prime.*

If the order of i were four and i^2 were not in D we would have the case considered above and hence such a G is impossible. The assumption that i^2 is in D also leads to a contradiction since the product of i and a non-invariant operator of order 2 in D would be of order 4, and this product together with the cyclic subgroup of half the order of D would generate an associate dihedral group. That is the order of i can not exceed 2. It remains therefore to consider the G's which involve at most two invariant operators including the identity.

By means of the preceding theorems it is not difficult to complete the consideration of all these groups when H is abelian. Since G is not dihedral and does not contain more than two invariant operators the order of H cannot be of the form 2^α. As the Sylow subgroup of order 2^m, $m > 2$, in G would be dihedral it follows that G would be dihedral. That is, it is impossible to construct a non-dihedral G when H is abelian and G does not involve more than two invariant operators unless G is of odd order. In this case G may be the abelian group of order p^2 and of type $(1, 1)$, p being an odd prime.

When H is dihedral and its cyclic subgroup (C) is of order $2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_\lambda^{\alpha_\lambda}$ we may choose s in such a way that all the operators of G which are not in H are of order 4, provided $\lambda = 1$, $\alpha_0 = 0$, $\alpha_1 = 1$, and $p \equiv 1 \bmod 4$. This is the only possible type when s^2 is not invariant under C. If s were commutative with more than two operators of C without being commutative with all of them, the subgroup generated by s and C would not be dihedral. Hence s is either commutative with all the operators of C, or it transforms each one of them into its inverse. In the former case we could replace s by the product obtained by multiplying s into some non-invariant operator of order 2 in H. As the latter can be so selected that the order of this product is again of the form 2^β, it may be assumed that s is of order 2 and transforms each operator of C into its inverse.

If s were commutative with a non-invariant operator of order 2 in H the product of s into this operator would be invariant under G. This case was considered above. If s were not commutative with such an operator it would have to transform the non-invariant operators of order 2 in H into themselves multiplied by operators whose order is divisible by 2^{α_0}, $\alpha_0 > 0$. In this case G would involve operators of order $2^{\alpha_0+1}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_\lambda^{\alpha_\lambda}$ and hence would be dihedral. Hence H cannot be dihedral when s^2 is found in C. We have now considered all the possible cases and may express the results as follows: Every non-abelian subgroup of a dihedral group is dihedral. If G is both non-abelian and non-dihedral, and if every non-abelian subgroup of G is dihedral then G belongs to one of the following five types of groups:

1) The direct product of the octic group and a group of order p, p being any prime number,
2) The direct product of the dihedral group of order $2q$, q being an odd prime, and the group of order p,
3) The direct product of the dihedral group of order $4q$ and the group of order 2,

4) The group obtained by extending the abelian group of order p^2, $p>2$, and of type (1, 1) by means of an operator of order 2 which transforms each operator of this abelian group into its inverse.

5) The group of order $4q$, q being prime and $\equiv 1 \bmod 4$, which is contained in the holomorph of the group of order q.

The first three types contain invariant operators while there is no invariant operator besides the identity in either of the last two types. None of these groups contains invariant operators of order p^2; p being any prime. If any other group contains a non-abelian dihedral subgroup it is either dihedral or it involves non-abelian subgroups which are not dihedral.

Lines of Curvature of a Surface.

By J. Edmund Wright.

It is known that a surface is intrinsically definite if the two fundamental forms ds^2 and ds^2/ρ are given. With the usual notation we represent these by

$$E\,du^2 + 2F\,du\,dv + G\,dv^2$$

and

$$D\,du^2 + 2D'\,du\,dv + D''\,dv^2.$$

In these two forms E, F, G may be given arbitrary values, and then D, D', D'' are subject to three relations. If the parametric lines are lines of curvature F and D' are both zero. In this case D, D'' are subject to three relations. These are not satisfied together unless a certain relation exists among the coefficients E, G of the first fundamental form. Hence if the parametric curves are lines of curvature E and G must be subject to a general condition.

If this condition is satisfied, it appears that D and D'' are in general definite and therefore the surface is intrinsically determinate.

It appears that the condition is equivalent to two differential equations of the fifth order for E and G. In this paper we determine the two equations, and apply the discussion to the particular case where $E = 1$. The remainder of the paper is concerned with a similar consideration in the case of asymptotic lines.

The equations for D, D'' are [*]

$$\frac{DD''}{\sqrt{EG}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) = 0 \qquad (1)$$

$$\frac{\partial}{\partial v}\left(\frac{D}{\sqrt{E}}\right) = \frac{D''}{G}\frac{\partial\sqrt{E}}{\partial v}, \qquad (2)$$

$$\frac{\partial}{\partial u}\left(\frac{D''}{\sqrt{G}}\right) = \frac{D}{E}\frac{\partial\sqrt{G}}{\partial u}. \qquad (3)$$

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[*] See Bianchi–Lukat, p. 94.

We write $D/\sqrt{E}=X$, $D''/\sqrt{G}=Y$, $E=A^2$, $G=C^2$, $p=\frac{1}{A}\frac{\partial C}{\partial u}$, $q=\frac{1}{C}\frac{\partial A}{\partial v}$, and use suffixes to denote differentiations with respect to u and v. The equations now become

$$XY+p_1+q_2=0, \qquad X_2=qY, \qquad Y_1=pX. \qquad (4)\ (5)\ (6)$$

If we write $p_1+q_2=-P$ we have $\frac{\partial}{\partial v}X^2=2qP$, $\frac{\partial}{\partial u}Y^2=2pP$. Hence $X^2=2\int qP\,dv$, $Y^2=2\int pP\,du$, where an arbitrary function enters through each of the integrals, and

$$P^2=4\int pP\,du\int qP\,dv. \qquad (7)$$

It is clear that, if this relation is satisfied, the equations are compatible, and therefore a surface exists with the given element of length, for which the parametric lines are lines of curvature.

We now write $P^2=a$, $2qP=b$, $2pP=c$, $X^2=\xi$; $Y^2=\eta$ and our equations become $\xi\eta=a$, $\xi_2=b$, $\eta_1=c$.

Consider first the case in which a, b, c, are all different from zero; ξ must satisfy the two equations $\frac{\partial}{\partial u}\left(\frac{a}{\xi}\right)=c$, $\frac{\partial\xi}{\partial v}=b$. Hence

$$\frac{a_1}{c}\xi-\xi^2-\frac{a}{c}\xi_1=0. \qquad (8)$$

If we differentiate this with respect to v we get

$$(a_1 b-ab_1)\,c+(a_{12}c-a_1 c_2-2bc^2)\,\xi+(ac_2-a_2 c)\,\xi_1=0. \qquad (9)$$

Now if $(a_1 b-ab_1)$ and $(ac_2-a_2 c)$ are both zero, $a_{12}c-a_1 c_2-2bc^2=0$ since $\xi\neq 0$.

If $ac_2-a_2 c$ only is zero, (9) determines ξ uniquely and then η is uniquely determined. In any case (8) and (9) determine ξ as the root of a quadratic equation, unless (9) is nugatory, and ξ_1 is determinate. We assume for the present that $ac_2-a_2 c\neq 0$. These equations give $\xi=\alpha$, say, and $\xi_1=\beta$. Also $\xi_2=b$. Hence we have two necessary and sufficient conditions for coexistence, namely $\alpha_2=b$, $\alpha_1=\beta$. These two conditions are of the third order in a, b, c, or of the fifth order in A and C. It is obvious that if they are satisfied, values of ξ and η can be determined to satisfy our system of equations, and further, there are only two possible sets of solutions.

We now show that in general there is only one such set; in fact, divide (9) by $ac_2 - a_2 c$ and differentiate again with respect to v. We get

$$\frac{(a_1 b - ab_1)(a_{22} c - ac_{22})c}{(ac_2 - a_2 c)^2} + \frac{2a_{12} bc - acb_{12} - 2b^2 c^2 + c(a_1 b_2 - 2a_2 b_1)}{ac_2 - a_2 c}$$
$$+ \frac{\partial}{\partial v}\left(\frac{a_{12} c - a_1 c_2 - 2bc^2}{ac_2 - a_2 c}\right)\xi = 0. \qquad (10)$$

If the coefficient of ξ in this equation is not zero, ξ is determined uniquely, and hence there is only one set. If this coefficient is zero we form the corresponding set of equations with η instead of ξ, and again we see that ξ and η are uniquely determinate unless

$$\frac{\partial}{\partial u}\left(\frac{a_{12} b - a_2 b_1 - 2b^2 c}{ab_1 - a_1 b}\right) = 0.$$

Hence unless both the expressions named are zero the surface having the given linear element, and the parametric lines as lines of curvature, is intrinsically determinate. Looking away from these particular cases we see that the two general equations of condition may be determined by eliminating ξ, ξ_1, from (8), (9), (10), and η, η_2, from the corresponding equations in η. The two equations thus obtained are obviously of the fifth order in A, C, and they are clearly independent, for fifth derivatives enter only through a_{123} in the first, and through a_{112} in the second.

The only case where ξ and η are not determinate arises if (9) is nugatory. We thus have the results:

(i) If the linear element of a surface is given by $ds^2 = A^2 du^2 + C^2 dv^2$, and if the parametric curves are lines of curvature, the surface in general is intrinsically uniquely determinate, and A and C must satisfy two equations involving their fifth derivatives. These two equations of condition are necessary and sufficient.

(ii) In certain cases there may be two surfaces satisfying the given conditions.

(iii) In certain cases there may be an infinite number of such surfaces.

We shall now enquire more closely into the surfaces of classes (ii) and (iii). The equations arising from the identical vanishing of (10) and its correspondent are rather cumbrous, so we consider the question again *ab initio*. We assume, in fact, that the equations $\frac{\partial}{\partial u}\left(\frac{a}{\xi}\right) = c$ and $\frac{\partial \xi}{\partial v} = b$ have two different solutions,

ξ and ξ'. This is equivalent to the case arising from the identical vanishing of (10) and the corresponding equation, with the exception that we must treat independently the case when the quadratic for ξ is a perfect square. We deduce at once that $\xi - \xi' = U$, $a\frac{\xi - \xi'}{\xi\xi'} = V$, when U and V are functions of u and v alone in that order. We still assume that a, b, c are different from zero, and therefore, since $\xi \neq \xi'$, U and V are both different from zero. We now take as new independent variables $\int \sqrt{U}\, du$ and $\int \sqrt{V}\, dv$, and we have $\xi - \xi' = 1$, $\xi\xi' = a$. Hence from the quadratic for ξ we get

$$\frac{a_1 b - ab_1}{a_2 c - ac_2} = 1. \tag{11}$$

$$\left\{\frac{a_{12} c - a_1 c_2 - 2bc^2}{ac_2 - a_2 c} + \frac{a_1}{a}\right\} \frac{a}{c} = \xi + \xi'.$$

Also if we substitute in turn ξ and ξ' in (9) and substract, we deduce

$$a_{12} c - a_1 c_2 - 2bc^2 = 0. \tag{12}$$

Hence $\xi + \xi' = \frac{a_1}{c}$. By substituting the values of the various coefficients in (9) we obtain $\xi_1 = c$, and therefore $c_2 = b_1$. Hence $c = \bar{\omega}_1$, $b = \bar{\omega}_2$, where $\bar{\omega}$ is a function of u and v. Also (11) becomes $a_1 \bar{\omega}_2 - a_2 \bar{\omega}_1 = 0$, and therefore $a = f(\bar{\omega})$. With these values the quadratic for ξ is $f'(\bar{\omega})\,\xi - \xi^2 - f(\bar{\omega}) = 0$, and (12) becomes $f'' - 2 = 0$.

$f(\bar{\omega})$ must therefore satisfy the two equations $f'' - 2 = 0$ and $1 = (\xi - \xi')^2 = f'^2 - 4f$. The first is seen to be a consequence of the second, and $f = (\bar{\omega} + \text{const.})^2 - \frac{1}{4}$. Our equations are now all satisfied and if we incorporate the constant into $\bar{\omega}$ we have

$$a = \bar{\omega}^2 - \tfrac{1}{4}, \quad b = \bar{\omega}_2, \quad c = \bar{\omega}_1;$$

$$\xi = \bar{\omega} + \tfrac{1}{2}, \quad \xi' = \bar{\omega} - \tfrac{1}{2}, \quad \eta = \bar{\omega} - \tfrac{1}{2}, \quad \eta' = \bar{\omega} + \tfrac{1}{2}.$$

We still have the relations

$$a = (p_1 + q_2)^2, \quad b = -2q(p_1 + q_2), \quad c = -2p(p_1 + q_2).$$

These give $p_1 + q_2 = \sqrt{\bar{\omega}^2 - \frac{1}{4}}$

$$p = -\frac{\bar{\omega}_1}{2\sqrt{\bar{\omega}^2 - \frac{1}{4}}}, \quad q = \frac{-\bar{\omega}_2}{2\sqrt{\bar{\omega}^2 - \frac{1}{4}}}$$

Hence

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{\bar{\omega}^2 - \frac{1}{4}}}\frac{\partial \bar{\omega}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{\bar{\omega}^2 - \frac{1}{4}}}\frac{\partial \bar{\omega}}{\partial v}\right) + 2\sqrt{\bar{\omega}^2 - \tfrac{1}{4}} = 0.$$

We write $\overline{\omega} = \frac{1}{2}\cosh\omega$ and this becomes

$$\frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} + \sinh\omega = 0. \tag{13}$$

In this investigation no distinction has been made between real and imaginary quantities, but the connection between surfaces with the same spherical representation as spherical or pseudospherical surfaces is immediate.*

We now consider the case when the quadratic for ξ has equal roots, and the equations of type (10) are nugatory. We have $(a_{12}c - a_1c_2 - 2bc^2)/(ac_2 - a_2c)$ equal to a function of u only. Let this function be $\frac{1}{U}\frac{dU}{du}$, and choose $\int \sqrt{U}\,du$ as the variable u. Similarly choose the new variable v. We now have

$$a_{12}c - a_1c_2 - 2bc^2 = 0,$$
$$a_{12}b - a_2b_1 - 2b^2c = 0,$$

and there is still a constant factor at our disposal in each of the variables u, v. From these two equation we deduce

$$\frac{a_1b}{a_2c} = \frac{b_1}{c_2},$$

and therefore each of these fractions is equal to

$$\frac{a_1b - ab_1}{a_2c - ac_2}.$$

If we substitute these values in (9) we get $\xi_1 = \frac{a_1}{a_2}b$, and therefore $\xi_1a_2 = \xi_2a_1$. ξ is therefore a function of a only, say $f(a)$. The quadratic for ξ is

$$\frac{a_1}{c}\xi - \xi^2 = \frac{aa_1b}{a_2c},$$

and since this is a perfect square we have

$$a_1a_2 = 4abc, \quad \xi = \frac{1}{2}\frac{a_1}{c}.$$

Now

$$\xi = f(a), \ \xi_2 = b, \ \frac{\partial}{\partial v}\left(\frac{a}{\xi}\right) = c,$$

and therefore

$$f'(a)a_1 = b, \left(\frac{1}{f} - \frac{af'}{f^2}\right)a_2 = c.$$

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* See Bianchi-Lukat, p. 472. The relation between the two surfaces corresponds to Hazzidaki's transformation. Eisenhart, Amer. Journal, Vol. 27, pp. 113–172, and Amer. Journal, Vol. 28, pp. 47–70.

Hence

$$\left(\frac{f'}{f} - a\frac{f'^2}{f^2}\right)a_1 a_2 = bc,$$

or

$$4\,a\left(\frac{f'}{f} - a\frac{f'^2}{f^2}\right) = 1.$$

We thus have $2af' - f = 0$, or $f = k\sqrt{a}$, where k is a constant. By choosing appropriately the constant factors at our disposal in u and v, we make $k = 1$, and we now have $\xi = \sqrt{a} = \eta$.

Further $\xi_2 = b$, $\eta_1 = c$, and hence

$$b = \frac{\partial\sqrt{a}}{\partial v}. \quad c = \frac{\partial\sqrt{a}}{\partial u}.$$

If we put $a = \bar{\omega}^2$ we have

$$a = \bar{\omega}^2, \;\; b = \bar{\omega}_2, \;\; c = \bar{\omega}_1$$
$$\xi = \bar{\omega}, \;\; \eta = \bar{\omega}.$$

Hence $p_1 + q_2 = \bar{\omega}$, $p = -\frac{1}{2}\frac{\bar{\omega}_1}{\bar{\omega}}$, $q = -\frac{1}{2}\frac{\bar{\omega}_2}{\bar{\omega}}$, and the equation for ω is

$$\frac{\partial}{\partial u}\left(\frac{1}{\bar{\omega}}\,\frac{\partial\bar{\omega}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\bar{\omega}}\,\frac{\partial\bar{\omega}}{\partial v}\right) + 2\,\bar{\omega} = 0;$$

or if we put $-2\,\bar{\omega} = e^{\omega}$

$$\frac{\partial^2\omega}{\partial u^2} + \frac{\partial^2\omega}{\partial v^2} = e^{\omega}. \tag{14}$$

The spherical representation for this class of surfaces is

$$d\sigma^2 = \bar{\omega}(du^2 + dv^2).^*$$

We now consider (iii), for which the quadratics for ξ and η are nugatory, and a, b, c are none of them zero. In the first place suppose $ac_2 - a_2 c = 0$, and that the remaining coefficients in (9) do not vanish. We have $\frac{e}{a} = U$, and by taking $\int U du$ as a new variable u, we have $c = a$.

The equations for ξ become

$$\frac{a_1}{a}\xi - \xi^2 = \xi_1.$$

$$(a_{12}\,a - a_1 a_2 - 2a^2 b)\,\xi = a\,(ab_1 - a_1 b),$$

and we have, as before, a unique value of ξ, and two equations of condition between a and b. There is a further limitation since $c = a$. This involves $p_1 + q_2 + 2p = 0$, and therefore

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* In this connection see Bianchi p. 185 sqq.

$$\frac{\partial}{\partial u}\left(e^{2u} p\right) + \frac{\partial}{\partial v}\left(e^{2u} q\right) = 0;$$

hence if ϕ is a certain function of u and v we have

$$e^{2u} p = \frac{\partial \phi}{\partial v}, \quad e^{2u} q = -\frac{\partial \phi}{\partial u},$$

and

$$a = \quad 4e^{-4u}\left(\frac{\partial \phi}{\partial v}\right)^2$$

$$b = -\ 4e^{-4u}\frac{\partial \phi}{\partial u}\,\frac{\partial \phi}{\partial v}.$$

We now suppose that the quadratic for ξ is nugatory. We have $a = c$, and $a_1 b - a b_1 = 0$. Hence by suitable choice of the variable v we may make $a = b$.

In addition, (9) shows that we must have $a a_{12} - a_1 a_2 - 2a^3 = 0$. If this condition is satisfied the equations $\xi\eta = a$, $\xi_2 = a$, $\eta_1 = a$ are compatible, but we must also have

$$-(p_1 + q_2) = 2p = 2q = \sqrt{a}.$$

Hence a must also satisfy

$$\frac{\partial}{\partial u}(\sqrt{a}) + \frac{\partial}{\partial v}(\sqrt{a}) + 2\sqrt{a} = 0.$$

From this equation $\sqrt{a} = e^{-2u} F(u - v)$, where F is an arbitrary function of its argument. But this value of a cannot satisfy the first condition for any value of F. Hence there are no surfaces of the kind sought.

We now consider the possibility hitherto neglected, namely that one of the quantities a, b, c is zero. Going back to the equations (4), (5), (6), we see that the condition requires either p or q to be zero. We assume $q = 0$ and we have

$$XY + p_1 = 0, \quad X_2 = 0, \quad Y_1 = pX,$$

together with $\frac{\partial A}{\partial v} = 0$. Hence A is a function of u only, and we may by suitable choice of u, make it unity. We have now

$$ds^2 = du^2 + C^2\,dv^2,$$

and $p = C_1$. The equations are readily solved; there are two possibilities accordinging as X is or is not zero. In the first case the surface is of zero curvature, and $ds^2 = du^2 + (au + b)^2 dv^2$, where a and b are functions of v only.

In the second case let $X = x'$, where x' is the first derivative of a function x of u only. We now have $X = x'$, $C_1 = V_1 \cos x + V_2 \sin x$, $Y = V_1 \sin x - V_2 \cos x$, where V_1 and V_2 are functions of v only.

The conditions C must satisfy, in order that it may be thrown into the given form, are

$$\begin{vmatrix} C_1 & C_{11} & C_{111} \\ C_{12} & C_{112} & C_{1112} \\ C_{122} & C_{1122} & C_{11122} \end{vmatrix} = 0,$$

and

$$\frac{\partial}{\partial u}\left[\frac{C_{11}\, C_{1112} - C_{112}\, C_{111}}{(C_1\, C_{112} - C_{11}\, C_{12})^3}\right] = 0.$$

The spherical representation is given by

$$d\sigma^2 = x'^2\, du^2 + (V_1 \sin x - V_2 \cos x)^2\, dv^2.$$

It readily follows that the surface is that swept out by a curve fixed in a given plane when the plane rolls on a developable surface.*

We now consider the case of a surface with linear element

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2,$$

for which the parametric lines are asymptotic. We have $D = D'' = 0$, and†

$$\frac{\partial}{\partial u}\frac{D'}{\sqrt{EG - F^2}} = -2\begin{Bmatrix}1\ 2\\2\end{Bmatrix}\frac{D'}{\sqrt{EG - F^2}}$$

$$\frac{\partial}{\partial v}\frac{D'}{\sqrt{EG - F^2}} = -2\begin{Bmatrix}1\ 2\\1\end{Bmatrix}\frac{D'}{\sqrt{EG - F^2}}.$$

Also if K is the total curvature

$$\frac{D'^2}{EG - F^2} = -K.$$

Hence E, F, G must be subject to the two conditions

$$\frac{\partial}{\partial u}\log K = -4\begin{Bmatrix}1\ 2\\2\end{Bmatrix}$$

$$\frac{\partial}{\partial v}\log K = -4\begin{Bmatrix}1\ 2\\1\end{Bmatrix}.$$

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* See also Bianchi-Lukat, p. 166.

† Bianchi-Lukat, p. 92.

These conditions are seen to be equivalent to two, one of the second and the other of the third in derivatives of E, F, G.

As a particular example, consider the case in which the asymptotic lines cut at right angles. The equations of condition give immediately

$$KE^2 = U, \qquad KG^2 = V,$$

where U is a function of u only, V is a function of V only. By appropriate choice of u and v we make $KE^2 = KG^2 = 1$.

We also have the additional equation

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = e^{\frac{1}{2}V},$$

where $K = e^V$.

Hence $$\sqrt{K} = \frac{4f'(u+iv)\,\phi'(u-iv)}{[f(u+iv)+\phi(u-iv)]^2}.$$

Also $H = 0$, and the spherical representation is given by $d\sigma^2 = -\sqrt{K}(du^2+dv^2)$. We note that since $ds^2 = \frac{1}{\sqrt{K}}(du^2+dv^2)$ the spherical representation is conformal, and the general properties of minima surfaces may be readily obtained from this point of view.

Suppose next that only one set of parametric curves, say that for which $v =$ constant, is asymptotic. In this case $D = 0$, and

$$\frac{D'}{\sqrt{EG-F^2}} = \sqrt{-K}.$$

In addition we must have

$$\frac{\partial}{\partial u}\sqrt{-K} = -2\left\{ {1\;2 \atop 2} \right\}\sqrt{-K} + \left\{ {1\;1 \atop 2} \right\}\frac{D''}{\sqrt{EG-F^2}},$$

$$\frac{\partial}{\partial u}\left(\frac{D''}{\sqrt{EG-F^2}}\right) - \frac{\partial}{\partial v}\sqrt{-K} =$$
$$2\left\{ {1\;2 \atop 1} \right\}\sqrt{-K} + \left\{ {1\;1 \atop 1} \right\}\frac{D''}{\sqrt{EG-F^2}}.$$

There is thus one equation of condition, of the fourth order, among the quantities E, F, G, and their derivatives, and this is given by substituting the value of D'' given by the first in the second of the above two equations. If this condition is satisfied D' and D'' are determinate and therefore the surface also is intrinsically determinate. As a particular case we readily deduce that if $E = 1$, $F = 0$, G must equal $au^2 + bu + c$, where a, b, c are three arbitrary functions of v only. This is of course the case of skew surfaces.

The general question as to what conditions are necessary in order that the parametric curves may be conjugates for some surface with a given linear element may also be treated in the same manner as that connected with the lines of curvature. We have to consider the possibility of coexistence of three equations

$$\xi\eta = a, \quad \xi_2 = \alpha\xi + \beta, \quad \eta_1 = \gamma\eta + \delta,$$

when a, α, β, γ, δ are known functions of u, v, we find that there must exist two relations of the fifth order among E, F, G and their derivatives.

BRYN MAWR, *April* 1906.

The Ovals of the Plane Sextic Curve.

By J. Edmund Wright.

It is known that the plane sextic without double points has at most eleven separate ovals, and Hilbert has stated, though without proof, that these cannot all lie external to one another. No proof of this theorem has in fact yet been given, though Miss Ragsdale* has shown that a sextic with the maximum number of ovals, all external, cannot be obtained by the ordinary processes whereby curves with the maximum number of circuits are derived. This paper gives a proof of the theorem.

Let it be assumed that $u = 0$ is a sextic with eleven external ovals all lying in the finite part of the plane, and suppose that u is chosen to be positive at infinity, then u is negative inside each oval, and positive outside. Now consider $u + c = 0$ where c is a positive constant. As c increases from zero, each oval gradually shrinks up. This process of shrinking continues either until one of the ovals reduces to an isolated point, or until an oval meets itself again, thus giving an ordinary node on that oval. This last case is however impossible, for if it were possible a slight further increase in c would give a sextic with twelve ovals. A sextic, even though reducible, cannot have twelve ovals. Let then $u + c_1 \equiv v = 0$ be the sextic with one isolated point and ten external ovals, no one of which contains this point. Now let A and B be two real straight lines through the point, and consider the sextic $v + k(A^2 + B^2) = 0$ where k is a positive constant. The particular point obtained, as k increases from zero, remains an isolated point on the sextic, and again the ovals shrink up until as before one reduces to an isolated point. Let k_1 be the value of k for this isolated point, and consider the sextic $w \equiv v + k_1(A^2 + B^2) = 0$. This sextic has nine external ovals, and two isolated points both lying outside all the ovals. Through the two isolated points may be described three real conics with no other common points, say L, M, N. Then, as before, we consider the sextic

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* American Journal of Mathematics (1906). *On the Arrangement of the real Branches of Plane Algebraic Curves.*

$$w + k\,(L^2 + M^2 + N^2) = 0,$$

and thus obtain a sextic with three isolated points. Now suppose that we have obtained a sextic with r isolated points, and $11 - r$ external ovals, and suppose r is less than eight, we can describe through the r points three real cubics C_1, C_2, C_3, which have no other common points, and therefore if $U = 0$ be the sextic, $U + k\,(C_1^2 + C_2^2 + C_3^2) = 0$ leads to a sextic with $r + 1$ isolated points and $10 - r$ ovals, all external to each other. If $r = 8$, two real cubics may be described through the eight points, and they have one other point common. This other point can lie inside one at most of the remaining ovals, and therefore $U + k\,(C_1^2 + C_2^2) = 0$ will give a sextic with two ovals and nine external isolated points. It is clear that only one cubic can be described through these points, for if there were more there would be an infinity of them, and one of these could be made to pass through an arbitrary point on the sextic. The cubic and the sextic would then intersect in 19 points, which is impossible. Let then $U = 0$ be the sextic with nine points and two ovals, and let $C = 0$ be the cubic through the nine points. Then $U + k\,C^2 = 0$ is a sextic with the nine isolated points as double points, and C has at most one even circuit. Hence, since C can never meet the sextic again, only one at most of the two ovals of U can contain any part of C inside it,* and therefore, as before, by increasing k from zero a sextic can be obtained with ten isolated points and one oval external to the ten points. The sextic is now unicursal, and the theorem to be proved is that there can exist no unicursal sextic with ten isolated points and an external oval. The sextic thus reduces to a finite oval and ten external isolated points, each at a finite distance from the others and from the oval. Let $S = 0$ be this sextic.

Through eight of the points describe two cubics C_1, C_2 each passing through one of the other two points. Also let C_1 and C_2 be so chosen that the oval lies in the part of the plane where they are positive, and consider the sextic $S + k\,C_1\,C_2 = 0$. As k increases from zero the eight points remain fixed isolated points and the two remaining points become ovals lying in the part of the plane for which $C_1\,C_2$ is negative. Also the large oval shrinks up. Now suppose for any point on this oval C_1 and C_2 have the values d_1 and d_2 and $\sqrt{S_x^2 + S_y^2}$ the value p. Then if k increases from zero to a small positive value, the oval shrinks up a small amount, and the point considered moves inward a distance $k\frac{d_1 d_2}{p}$. Also suppose at the 9th point (on C_1), C_2 is δ_2, C_1 is $ax + by + \ldots.$ and S is $\alpha x^2 + 2\beta xy + \gamma y^2 + \ldots,$ where $x = 0$ $y = 0$ is the point considered. The small oval is practically the ellipse $\alpha x^2 + 2\beta xy + \gamma y^2 + k\delta_2(ax + by) = 0$, so that this ellipse has dimensions proportional to k, and it moves towards the oval only when a branch of C_1 or C_2 separates it from the oval. Hence if the oval is at a finite distance from C_1 and C_2 we can thus obtain a sextic with eight isolated points and three external ovals, and the first oval is made smaller than before. We now proceed as before to shrink up the three ovals and so obtain again ten points and one oval. If the original oval should be one of the two that reduce to points, we can arrange the second process so as to stop at a smaller value of k, and thus by choosing k properly have the three ovals all points. This would give a sextic with eleven isolated points. The process must be stopped if the sextic approaches to within a small distance of one of the cubics. If however a part of the oval is at a distance $<\varepsilon$ from say C_1 it must be possible to write $S = C_1 C' + \varepsilon S'$, where C' is a cubic and S' is a sextic with finite coefficients. But obviously a sextic such as that considered cannot be obtained from any two cubics and the only possibility is that C_1 consists of a straight line and a conic. If C_1 consists of a straight line and a conic, then there remains to be considered the possibility $S = PQ + \varepsilon S'$ where P is a conic and Q is a quartic, and the only possibility is that Q is a perfect square passing through the ten isolated points. This however is impossible since a conic cannot pass through more than six of the *d. p. s* of a sextic. Hence the oval is always at a finite distance away from the cubics and thus the process can be continued until the ovals reduce to eleven isolated *d. p. s.* There remains yet one possibility. The two *d. p. s* which are varied might run together. Suppose that the process is continued until the two variable points are at a very small distance apart, and then take one of these as one of the eight fixed points, allowing one of the first eight to be movable, and continue the process until these are a small distance apart if before this time the oval of the sextic cannot be reduced to a point. We can in this way keep seven points fixed and continue our process until the three points come close together. Now these points must in the limit form a point equivalent to three ordinary *d. p. s,* and it cannot be an ordinary triple point, for then a real branch of the sextic would necessarily pass through it.

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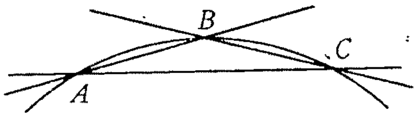
* If an oval of the cubic lies inside an oval of the sextic, that oval cannot shrink up.

It must therefore be of the type known as an oscnode, and a cubic passing through the three when they are at a small distance apart will have an ordinary point at the place. Describe the cubic through the three and six of the seven fixed points. Let the three special ones be A, B, C and consider the three cubics through BC, CA, AB, and the other seven points. Suppose that C_1 is the cubic through the three ABC, and C_2 is, say, that through BC. Then A and the point not on C_1 turn into small ovals which may possibly approach each other. But it is easily seen that the cubics BC and CA are close to each other, and therefore if we take C_2 as the cubic CA the ovals derived from B and the point not on C_1 will be separated from one another by C_2. They therefore cannot approach very close to each other without first separating. They cannot approach across C_2. If the branch between them be the even branch of C_2 they can never approach. If it be the odd branch they can only opproach across ∞y. Hence through the whole process we may always hold six points fixed and can continue indefinitely the alternate process outlined, always arranging that the four movable points shall not run together, and we finally obtain a sextic consisting of eleven isolated points. Such a sextic would, however, be reducible, and it may easily be verified that no such reducible sextic exists. Hence the theorem is proved that *a sextic cannot consist of eleven external ovals.*



I shall show in a later paper that the only possible arrangements for the ovals of a sextic are one internal and ten external, or ten internal and one external.

BRYN MAWR, *March* 1907.

On Twisted Quintic Curves.

BY E. C. COLPITTS.

INTRODUCTION.

The method of depicting on a conic the rational R_5 that lies on a hyperboloid is the same as that used by Emil Weyr for the rational quartic curve.[*] The theory of biquadratic involution used in this paper is taken from two other articles by Weyr.[†]

The following is a list of the papers that have been published on twisted quintic curves:

(1). On the rational curves.

E. Bertini. Su le curve gobbe razionali del 5° ordine. *Collect. math. in memoriam D. Chelini*, Mediolani, 1881, pp. 313–326.

L. Berzolari. Su la curva gobba razionale del quint'ordine. *Atti della R. Acc. dei Lincei*, Rome, 1894, pp. 305–341.

G. K. Nugteren. Rationale Ruimtekrommen van de vijfde orde. *Diss., Utrecht*, 1901, 67 pp.

Alfons Schmitz. Ueber eine bemerkenswerte Raumkurve fünfter Ordnung. *Diss., Munich*, 1887, 58 pp.

(2). On the curves of genus one.

Emil Weyr. Ueber Raumcurven fünfter Ordnung vom Geschlechte Eins. *Wiener Berichte*, vol. 90, 1884, pp. 206–225; vol. 92, 1885, pp. 498–523; vol. 97, 1888, pp. 592–617.

D. Montesano. Su la curva gobba di 5° ordine e di genere 1. *Acc. Napoli*, vol. 27, 1888, pp. 181–188.

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[*] Ueber die Abbildung einer rationalen Raumcurve vierter Ordnung auf einen Kegelschnitt. *Wiener Berichte*, vol. 72, 1875.

[†] Ueber biquadratische Involutionen erster Stufe. *Wiener Berichte*, vol. 88, 1881, pp. 300–320.

Ueber Ausartungen biquadratischer Involutionen. *Wiener Berichte*, vol. 88, pp. 807–827.

(3). On the curves of genus two.

H. E. Timerding. Ueber eine Raumcurve fünfter Ordnung. *Crelle*, vol. 123, 1901, pp. 284–311.

I have made use of the paper by Nugteren in the discussion of the rational quintic curves, and those by Weyr and Montesano for the curves of genus one.

I have treated the rational quintics by writing the equations in terms of a parameter. The conditions for singularities on the rational curve are found in this way. The (1, 4) quintic lying on a hyperboloid is, however, studied by depicting it on a conic. I have thus found twelve curves of this type besides the non-singular case. At the close of the chapter dealing with rational quintics is a list of the quintic curves that belong to a linear complex. The quintic curve of genus one on a cubic surface F_3 is depicted on a plane, and the number of times that each line of F_3 cuts R_5 is determined. The same thing is done for the case when F_3 has a double point. It is then shown that the trisecants of R_5 of genus one belong to a linear complex and that the pole of any plane lies on the conic determined by the five points in which the plane cuts R_5. By means of the method used by Clebsch I have written the equation of the plane projection of the quintic curve of genus two and found the maximum number of inflexional generators of the curve.

CHAPTER I.

ON THE RATIONAL QUINTIC CURVES.

§1. *Proof for the Existence of at Least one Quadrisecant.*

The rational quintic curves are divided into two classes; every quintic R_5^0 of the first kind is the partial intersection of two surfaces of the third order, the residual curve consisting of a curve R_3 and a line which does not cut this curve; or it may be the partial intersection of two ruled surfaces of the third order which have the same double directrix. The rational R_5^0 of the second kind is the partial intersection of a quadric surface F_2 with a ruled surface of the fourth order which has a quadrisecant S_4 of R_5^0 for a triple line; or it is the intersection of a quadric surface F_2 with a quartic surface F_4, the residual intersection being three skew lines.

Besides the non-singular rational quintic curves ($h = 6$), there are various other forms, viz.:

$h = 5,\ H = 1$; $h = 5,\ \beta = 1$; $h = 4,\ H = 2$; $h = 4,\ H = 1,\ \beta = 1$;

$h = 4$, $\beta = 2$; $h = 3$, with a triple point. A triple point may have three distinct tangents; or all three coincident.* For the Cayley numbers of the quintic I use the symbols and relations as given by E. Pascal, Repertorium der höheren Mathematik, vol. 2, pp. 225–6.

A rational quintic curve R_5 may be expressed by

$$\rho x = \sum_{i=0}^{5} a_i t^{5-i}, \quad \rho y = \sum_{i=0}^{5} b_i t^{5-i}, \quad \rho z = \sum_{i=0}^{5} c_i t^{5-i}, \quad \rho w = \sum_{i=0}^{5} d_i t^{5-i}.$$

If R_5 be cut by a plane $Ax + By + Cz + Dw = 0$ the parameters of the points of intersection are defined by $\sum_{i=0}^{5} (Aa_i + Bb_i + Cc_i + Dd_i)\, t^{5-i} = 0$, say t_1, t_2, t_3, t_4, t_5. If $\Sigma t_1 = \Sigma_1$, $\Sigma t_1 t_2 = \Sigma_2$, etc., we have

$$(Aa_0)\,\Sigma_i + (-1)^{i-1}\,(Aa_i) = 0, \; i = 1, \ldots, 5.$$

where $(Aa_i) = Aa_i + Bb_i + Cc_i + Dd_i$. Eliminate A, B, C, D between these equations. The result will be the conditions that the points $t_1 \ldots t_5$ are coplanar. After various reductions one condition may be written,

$$(a_0 b_1 c_2 d_3)\,\Sigma_4 + (a_0 b_1 c_2 d_4)\,\Sigma_3 + (a_0 b_1 c_3 d_4)\,\Sigma_2 + (a_0 b_2 c_3 d_4)\,\Sigma_1 + (a_1 b_2 c_3 d_4) = 0$$

or say,

I. $$P_1 \Sigma_4 + P_2 \Sigma_3 + P_3 \Sigma_2 + P_4 \Sigma_1 + P_5 = 0.$$

The remaining condition is

$$(a_0 b_1 c_2 d_3)\,\Sigma_5 - (a_0 b_1 c_2 d_5)\,\Sigma_3 - (a_0 b_1 c_3 d_5)\,\Sigma_2 - (a_0 b_2 c_3 d_5)\,\Sigma_1 - (a_1 b_2 c_3 d_5) = 0$$

or

I′. $$q_0 \Sigma_5 + q_2 \Sigma_3 + q_3 \Sigma_2 + q_4 \Sigma_1 + q_5 = 0.$$

If definite values are given to t_1, t_2, t_3, t_4, then t_5 must satisfy the equations.

$$(P_1\sigma_4 + P_2\sigma_3 + P_3\sigma_2 + P_4\sigma_1 + P_5) + (P_1\sigma_3 + P_2\sigma_2 + P_3\sigma_1 + P_4)\,t_5 = 0.$$
$$(q_2\sigma_3 + q_3\sigma_2 + q_4\sigma_1 + q_5) + (q_0\sigma_4 + q_2\sigma_2 + q_3\sigma_1 + q_4)\,t_5 = 0.$$

where σ_i denotes the fundamental symmetric functions of t_1, t_2, t_3, t_4 of weight i. The value of t_5 is indeterminate if

$$\begin{aligned} P_1\sigma_4 + P_2\sigma_3 + P_3\sigma_2 + P_4\sigma_1 + P_5 &= 0. \\ P_1\sigma_3 + P_2\sigma_2 + P_3\sigma_1 + P_4 &= 0. \\ q_2\sigma_3 + q_3\sigma_2 + q_4\sigma_1 + q_5 &= 0. \\ q_0\sigma_4 \qquad + q_2\sigma_2 + q_3\sigma_1 + q_4 &= 0. \end{aligned}$$

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* Two coincident tangents can also exist, *e. g.*, $x = t^4(t-1)$, $y = t^3(t-1)^2$, $z = t^2(t-1)^3$, $w = \phi_5(t)$. (See Van der Vries: *Proc. Amer. Acad.*, vol. 38, p. 524, footnote).

These four equations are linear in t_1, t_2, t_3, t_4. Hence *every rational R_5 has one quadrisecant.* R_5 has ∞' quadrisecants if these equations are not independent, that is, if

$$\text{II.} \qquad \begin{vmatrix} P_1, & P_2, & P_3, & P_4 \\ & P_1, & P_2, & P_3 \\ & q_2, & q_3, & q_4 \\ q_0, & & q_2, & q_3 \end{vmatrix} = 0.$$

Synthetic proof: Establish a projectivity between the points of R_5^0 and any S_3, having the three points of intersection as corresponding points. Draw the lines connecting a point x on S_3 with its corresponding point x' on R_5. These lines will generate a ruled surface on which s_3 is a simple directrix. Every plane through S_3 cuts R_5 in two points x' not on the line, hence contains two generators. The line S_3 cannot be a generator since each point of R_5^0 upon it is self-correspondent, hence R_5^0 lies on an infinite number of cubic scrolls. One point x' lies on each generator, hence a plane through the double directrix will only cut the curve in one variable point, from which it follows that R_5^0 has a quadrisecant. If it has only one, every such ruled cubic has this same quadrisecant for double directrix: if it has more than one it lies on a hyperboloid.*

§ 2. *R_5 as the Intersection of Two Ruled Surfaces of the Third Order.*†

From the preceding theorem it is seen that every R_5^0 can be defined as the intersection of two ruled cubic surfaces, having a double line in common. The equations

$$U_3 + zU_2 + V_2 = 0, \quad U_3' + zU_2' + V_2' = 0,$$

in which U_i, V_2 are polynomials in x, y of degree i, 2, represent two ruled surfaces each having the line $x = 0$, $y = 0$, for double directrix. Their intersection is made up of the double directrix d_2 counted four times and a twisted quintic curve R_5 having d_2 for a quadrisecant.

Let the surface

$$(a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3) + z(b_0 x^2 + 2b_1 xy + b_2 y^2) \\ + (c_0 x^2 + 2c_1 xy + c_2 y^2) = 0.$$

be given. Any plane through the double directrix is $y = mx$. Substitute $y = mx$

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* Another proof that every rational R_5 has one quadrisecant is given by Bertini in *Collectanea Mathematica in Memoriam D. Chelini*, Mediolani, 1881.

† G. K. Nugteren, Rationale Ruimte krommen van de Vijfde Orde, Diss., Utrecht, 1901.

in the above equation. The resulting equation is

$$(a_0 x^3 + 3a_1 mx^3 + 3a_2 m^2 x^3 + a_3 m^3 x^3) + z(b_0 x^2 + 2b_1 mx^2 + b_2 m^2 x^2) \\ + (c_0 x^2 + 2c_1 mx^2 + c_2 m^2 x^2) = 0.$$

Besides $x^2 = 0$, we have

$$(a_0 x + 3a_1 mx + 3a_2 m^2 x + a_3 m^3 x) + z(b_0 + 2b_1 m + b_2 m^2) \\ + (c_0 + 2c_1 m + c_2 m^2) = 0$$

or
$$a_m^3 x + b_m^2 z + c_m^2 = 0.$$

The intersection of $y = mx$ with $x^2 = 0$ defines d_2; $y = mx$ and $a_m^3 x + b_m^2 z + c_m^2 = 0$ define a generator of the surface. We have the (1, 2) relation between z and m

$$z(b_0 + 2b_1 m + b_2 m^2) + (c_0 + 2c_1 m + c_2 m^2) = 0$$

when the planes $x = 0$, and $a_m^3 x + b_m^2 z + c_m^2 = 0$ intersect. Hence, through every point of d_2 pass two planes cutting out generators of F_3, or two generators pass through every point of d_2.

For $m = 0$, $z = -\frac{c_0}{b_0}$. The other value of m corresponding to $z = -\frac{c_0}{b_0}$ is $m = -\frac{2(b_0 c_1)}{(b_0 c_2)}$. The general equation of a plane through these two generators is $Ax + By + Cz + D = 0$. This plane contains two lines which are the intersections of the planes $y = mx$ and $a_m^3 x + b_m^2 z + c_m^2 = 0$, for $m = 0$ and $m = -\frac{2(b_0 c_1)}{(b_0 c_2)}$ or say, $= -2t$. If $m = 0$, we have $y = mx = 0$, and $a_0 x + b_0 z + c_0 = 0$.

Therefore $Ax + By + Cz + D = 0$, must be the same plane as $a_0 x + b_0 z + c_0 + B'y = 0$ (1) for some quantity B'. Similarly if $m = -2t$, we have the two planes $y + 2tx = 0$, and

$$(a_0 - 6a_1 t + 12a_2 t^2 - 8a_3 t^3)x + (b_0 - 4b_1 t + b_2 t^2)z + (c_0 - 4c_1 t + 4c_2 t^2) = 0.$$

As before, the plane $Ax + By + Cz + D = 0$ must coincide with the plane

$$(a_0 - 6a_1 t + 12a_2 t^2 - 8a_3 t^3)x + (b_0 - 4b_1 t + b_2 t^2)z \\ + (c_0 - 4c_1 t + 4c_2 t^2) + B''(y + 2tx) = 0. \qquad (2)$$

for some value B''. Since (1) and (2) represent the same plane.

$$\frac{a_0 - 6a_1 t + 12a_2 t^2 - 8a_3 t^3 + 2B' t}{a_0} = \frac{B''}{B'} = \frac{b_0 - 4b_1 t + 4b_2 t^2}{b_0} \\ = \frac{c_0 - 4c_1 t + 4c_2 t^2}{c_0}.$$

From these relations we find that

$$B' = \frac{c_0}{b_0} \frac{(3a_1 b_0 - 2a_0 b_1 - 6a_2 b_0 t + 2a_0 b_2 t + 4a_3 t^2)}{c_0 - 4c_1 t + 4c_2 t^2}.$$

If now we substitute for t its value $\frac{(b_0 c_1)}{(b_0 c_2)}$, we find

$$B' = \frac{3a_1 (b_0 c_2)^2 - 6a_2 (b_0 c_2)(b_0 c_1) - 2a_0 (b_0 c_2)(b_1 c_2) + 4a_3 (b_0 c_1)^2}{(b_0 c_2)^2 - 4 (b_0 c_1)(b_1 c_2)}.$$

Again, for $m = \infty$, $z = -\frac{c_2}{b_2}$. The other value of m is found to be $-\frac{1}{2}\frac{(b_0 c_2)}{(b_1 c_2)}$. A general plane through the two generators corresponding to $m = \infty$ and $m = -\frac{1}{2}\frac{(b_0 c_2)}{(b_1 c_2)} = -\frac{1}{2}v$, say, is $Ax + By + Cz + D = 0$. For $m = \infty$,

$$a_3 y + b_2 z + c_2 + A'x = 0 \tag{3}$$

represents the same plane. For $m = -\frac{v}{2}$, we have the two planes $y + \frac{v}{2}x = 0$,

$$\left(a_0 - \frac{3}{2}a_1 v + \frac{3}{4}a_2 v^2 - \frac{a_3}{8}v^3\right)x + \left(b_0 - b_1 v + \frac{b_2}{4}v^2\right)z + \left(c_0 - c_1 v + \frac{c_2}{4}v^2\right) = 0.$$

Any plane through their line of intersection is

$$\left(a_0 - \frac{3}{2}a_1 v + \frac{3}{4}a_2 v^2 - \frac{a_3}{8}v^3\right)x + \left(b_0 - b_1 v + \frac{b_2}{4}v^2\right)z$$
$$+ \left(c_0 - c_1 v + \frac{c_2}{4}v^2\right) + B''\left(y + \frac{v}{2}x\right) = 0. \tag{4}$$

If the planes (3) and (4) are identical

$$\frac{a_0 - \frac{3}{2}a_1 v + \frac{3}{4}a_2 v^2 - \frac{a_3}{8}v^3 + \frac{B''v}{2}}{A'} = \frac{B''}{a_3} = \frac{b_0 - b_1 v + \frac{b_2}{4}v^2}{b_2}$$
$$= \frac{c_0 - c_1 v + \frac{c_2}{4}v^2}{c_2}.$$

It follows that

$$A' = c_2 \frac{\left(a_0 - \frac{3}{2}a_1 v + \frac{3}{4}a_2 v^2 + \frac{1}{2}a_3 \frac{c_0 v}{c_2} - \frac{1}{2}a_3 \frac{c_1}{c_2} v^2\right)}{c_0 - c_1 v + \frac{c_2}{4}v^2}$$
$$= \frac{3a_2 (b_0 c_2)^2 - 6a_1 (b_0 c_2)(b_1 c_2) - 2a_3 (b_0 c_1)(b_0 c_2) + 4a_0 (b_1 c_2)^2}{(b_0 c_2)^2 - 4 (b_1 c_2)(b_0 c_1)}.$$

The intersection of the two planes

$$a_0 x + By + b_0 z + c_0 = 0$$
$$A'x + a_3 y + b_2 z + c_2 = 0$$

is the simple directrix d_1 of F_3. The coordinates of the point of intersection of d_1 with the xy plane are $z = 0$,

$$x = \frac{-a_3 c_0 + c_2 B}{a_0 a_3 - A'B}, \quad y = \frac{-a_0 c_2 + c_0 A'}{a_0 a_3 - A'B}.$$

Let $B = \frac{t_5}{n_4}$, $A' = \frac{t'_5}{n_4}$. Then $x = \frac{-a_3 c_0 n_4 + c_2 t_5}{a_0 a_3 n_4^2 - t_5 t'_5} n_4 = \frac{N}{D} n_4$.

$$N = -a_3 c_0 [(b_0 c_2)^2 - 4 (b_0 c_1)(b_1 c_2)] + c_2 [3a_1 (b_0 c_2)^2$$
$$- 6a_2 (b_0 c_2)(b_0 c_1) - 2a_0 (b_0 c_2)(b_1 c_2) + 4a_3 (b_0 c_1)^2].$$

But N is divisible by $(b_0 c_2)$. For we have

$$4a_3 (b_0 c_1) [(b_0 c_1) c_2 + (b_1 c_2) c_0]$$
$$= 4a_3 (b_0 c_1) [b_0 c_1 c_2 - b_1 c_0 c_2 + b_1 c_2 c_0 - b_2 c_1 c_0]$$
$$= 4a_3 (b_0 c_1)(b_0 c_2) c_1.$$

$$D = a_0 a_3 [(b_0 c_2)^2 - 4 (b_1 c_2)(b_0 c_1)]^2 - [(P (b_0 c_2) + 4a_3 (b_0 c_1)^2)$$
$$\times (Q (b_0 c_2) + 4a_0 (b_1 c_2)^2)]$$
$$= a_0 a_3 [(b_0 c_2)^4 - 8 (b_0 c_2)^2 (b_1 c_2)(b_0 c_1) + 16 (b_1 c_2)^2 (b_0 c_1)^2]$$
$$- [R (b_0 c_2) + 16\, a_0 a_3 (b_1 c_2)^2 (b_0 c_1)^2]$$
$$= a_0 a_3 [(b_0 c_2)^4 - 8 (b_0 c_2)^2 (b_1 c_2)(b_0 c_1)] - R (b_0 c_2).$$

Therefore, both N and D are divisible by $(b_0 c_2)$, and the numerator and the denominator of x are both of degree eight. Similarly it may be shown the numerator and denominator of y are also of degree eight.

Form the pencil of surfaces

$$(U_3 + zU_2 + V_2) + t(U'_3 + zU'_2 + V'_2) = 0,$$

all of which pass through R_5. Then for a we write $a + ta'$; for b, $b + tb'$; for c, $c + tc'$. The simple directrix meets the xy plane in the point $x = \frac{f_8(t)}{h_8(t)}$, $y = \frac{g_8(t)}{h_8(t)}$, $z = 0$. Hence the locus of the point in which d_1 cuts the xy plane is a rational curve of order eight. Since the double directrix d_2 has four points on R_5, each generator has only one point. Hence, d_1 is a trisecant of R_5. *The simple directrices of the surfaces of the pencil $F_3 + tF'_3$ are the trisecants of R_5; and the trisecants form a rational ruled surface of order eight.* Hence, R_5 lies on ∞^1

ruled cubic surfaces all of which have the line d_2 for double directrix, and each surface has a different trisecant of R_5 for simple directrix.

We will now find the number of Cayley cubic scrolls in this pencil of surfaces. The equation of the Cayley scroll can be put in the form

$$U_3 + (Px + qy)\,[(b_0 x + b_1 y)\,z + (c_0 x + c_1 y)] = 0.$$

If

$$U_3 + (b_0 x^2 + 2b_1 xy + b_2 y^2)\,z + (c_0 x^2 + 2c_1 xy + c_2 y^2) = 0.$$

and

$$U'_3 + (b'_0 x^2 + 2b'_1 xy + b'_2 y^2)\,z + (c'_0 x^2 + 2c'_1 xy + c'_2 y^2) = 0.$$

are the equations of two Cayley scrolls, then the terms $U_2 + tU'_2$ and $V_2 + tV'_2$ have a common factor, when we form the pencil $F_3 + tF'_3 = 0$; or

$$(b_0 + tb'_0)\left(\frac{x}{y}\right)^2 + 2\,(b_1 + tb'_1)\left(\frac{x}{y}\right) + (b_2 + tb'_2) = 0$$

$$(c_0 + tc'_0)\left(\frac{x}{y}\right)^2 + 2\,(c_1 + tc'_1)\left(\frac{x}{y}\right) + (c_2 + tc'_2) = 0$$

have a root $\frac{x}{y}$ in common. The resultant of these two equations is of order four in the coefficients. Hence, *R_5 lies on four Cayley scrolls.*

The number of Cayley scrolls can be found by another method. R_5 has only one quadrisecant. Take three of the points of S_4 on R_5 as corresponding to the same three points of R_5. This fixes the projectivity between the points of S_4 and of R_5. Any plane through S_4 cuts R_5 in one point. The lines joining such points to their corresponding points on S_4 generate a Cayley cubic scroll, for which S_4 is the directrix. The three coincidences can be chosen among the four points in four different ways. Hence, there are four Cayley scrolls on which R_5 lies.

It is evident that the following statement in Pascal's Repertorium der höheren Mathematik, Vol. 2, p. 266, must be incorrect; "Jede R_5^0 der ersten Species ist der theilweise Schnitt zweier Regelflächen 3^{ter} Ordnung, welche ausserdem eine Doppelgerade und zwei andere Gerade gemeinschaftlich haben, von denen die eine die Doppelgerade schneidet und die andere nicht."

§ 3. *Plane projection of R_5.*

It has been shown that the trisecants form a rational ruled surface F_8 of order eight. Since R_5 projects from any one of its points into a rational plane curve of order four, it follows that three trisecants can be drawn from every point of R_5. Hence R_5 is a triple curve on F_8. No other trisecants can be drawn through the four points in which S_4 cuts R_5; for if this were possible a plane through S_4 and S_3 would meet R_5 in six points. Hence, S_4 counts for six generators. The complete nodal curve of F_8 consists of the original R_5 counted three times and S_4 counted as six double generators.

From any point not on the curve, R_5 projects into a rational plane quintic C_5. The configuration of cusps and double points on C_5 depends on the position in space of the point of projection. From any point on S_4, R_5 projects into a C_5 with a fourfold point; from any point of F_8 into a C_5 with one triple point and three double points. The tangents of R_5 form a planar developable D_8 of order eight. The point of tangency projects into a cuspidal point. Hence, from an ordinary point an D_8, R_5 projects into a C_5 with one cusp and five double points; from a point on the double curve of D_8 into a C_5 with two cusps and four double points; from one of the triple points of D_8 into a C_5 having three cusps and three double points. The plane projection of R_5 from an arbitrary point in space has six double points.

§ 4. *Derivation of $F(t)$ and the Conditions for some Singularities.*

If from equations I, I′ we eliminate t_5, and let the remaining parameters all become equal we have.

$$\begin{aligned}F(t) = {} & P_1 q_0 t^8 + 4P_2 q_0 t^7 + (6P_3 q_0 - 10P_1 q_2) t^6 + (4P_4 q_0 - 20P_1 q_3) t^5 \\ & + (P_5 q_0 + 20P_3 q_2 - 20P_2 q_3 - 15P_1 q_4) t^4 + (20P_4 q_2 - 20P_2 q_4 - 4P_1 q_5) t^3 \\ & + (6P_5 q_2 + 10P_4 q_3 - 10P_3 q_4 - 6P_2 q_5) t^2 + 4(P_5 q_3 - P_3 q_5) t \\ & + (P_5 q_4 - P_4 q_5) = 0.\end{aligned}$$

The roots of $F(t)$ are the parameters of the points of contact of the stationary planes of R_5. Hence, *R_5 has eight stationary planes.*

Writing S_1 for $t_1 + t_2 + t_3$, S_2 for $t_1 t_2 +$ etc.; $S_3 = t_1 t_2 t_3$, the conditions that five points should be coplanar may be written

$$\begin{aligned}& P_1 [S_3 (t_4 + t_5) + S_2 t_4 t_5] + P_2 [S_3 + S_2 (t_4 + t_5) + S_1 t_4 t_5] \\ & \qquad + P_3 [S_2 + S_1 (t_4 + t_5) + t_4 t_5] + P_4 [S_1 + (t_4 + t_5)] + P_5 = 0.\end{aligned}$$

$$\begin{aligned}& q_0 S_3 t_4 t_5 + q_2 [S_3 + S_2 (t_4 + t_5) + S_1 t_4 t_5] + q_3 [S_2 + S_1 (t_4 + t_5) + t_4 t_5] \\ & \qquad + q_4 [S_1 + (t_4 + t_5)] + q_5 = 0.\end{aligned}$$

If t_1, t_2, t_3 are collinear they lie in a plane no matter what values are given to $(t_4 + t_5)$ or to $t_4 t_5$. Hence

$$\text{III.}\quad \frac{P_2 S_3 + P_3 S_2 + P_4 S_1 + P_5}{q_2 S_3 + q_3 S_2 + q_4 S_1 + q_5} = \frac{P_1 S_3 + P_2 S_2 + P_3 S_1 + P_4}{q_2 S_2 + q_3 S_1 + q_4} = \frac{P_1 S_2 + P_2 S_1 + P_3}{q_0 S_3 + q_2 S_1 + q_3}.$$

express the condition that three points of R_5 lie on a trisecant.

Let t_3 be given the fixed value t, and write T_1 for $t_1 + t_2$ and T_2 for $t_1 t_2$. These equations may then be written

$$\frac{(P_5 + P_4 T_1 + P_3 T_2) + (P_4 + P_3 T_1 + P_2 T_2)\,t}{(q_5 + q_4 T_1 + q_3 T_2) + (q_4 + q_3 T_1 + q_2 T_2)\,t} = \frac{(P_4 + P_3 T_1 + P_2 T_2) + (P_3 + P_2 T_1 + P_1 T_2)\,t}{(q_4 + q_3 T_1 + q_2 T_2) + (q_3 + q_2 T_1)\,t} = \frac{(P_3 + P_2 T_1 + P_1 T_2) + (P_2 + P_1 T_1)\,t}{(q_3 + q_2 T_1) + (q_2 + q_0 T_2)\,t}.$$

Put each of these fractions equal to K and eliminate T_1, T_2, 1, thus:

$$\begin{vmatrix} (P_5 + P_4 t) - K(q_5 + q_4 t), & (P_4 + P_3 t) - K(q_4 + q_3 t), & (P_3 + P_2 t) - K(q_3 + q_2 t) \\ \cdots\cdots\cdots\cdots, & \cdots\cdots\cdots\cdots, & \cdots\cdots\cdots\cdots \\ (P_3 + P_2 t) - K(q_3 + q_2 t), & \cdots\cdots\cdots\cdots, & \cdots\cdots\cdots\cdots \end{vmatrix} = 0.$$

This is a cubic equation in K; hence K has three values K_1, K_2, K_3. Put two of these equations each equal to K_i. This determines T_1 and T_2, and therefore, one pair of roots $t_1 t_2$ on a line with t. Hence: *Three trisecants can be drawn through every point of* R_5.

If $t_1 = t_2 = t_3$ the trisecant becomes a tangent at the point of inflexion. Hence, the condition for a linear inflexion may be found by putting $t_1 = t_2 = t_3$ in III. The resulting equations are

$$\frac{P_2 t^3 + 3P_3 t^2 + 3P_4 t + P_5}{q_2 t^3 + 3q_3 t^2 + 3q_4 t + q_5} = \frac{P_1 t^3 + 3P_2 t^2 + 3P_3 t + P_4}{3q_2 t^2 + 3q_3 t + q_4} = \frac{3P_1 t^2 + 3P_2 t + P_3}{q_0 t^3 + 3q_2 t + q_3}.$$

This gives only two independent relations which may be written $\sum_{i=0}^{6} V_i t^i = 0$, $\sum_{i=0}^{6} S_i t^i = 0$, where the coefficients V_i, S_i are of order eight in the coefficients a, b, c, d. If R_5 have a linear inflexion these two sextics must have a common root.

The resultant IV of degree ninety-six expresses the condition for a linear inflexion on R_5.

When two stationary planes become coincident, $F(t) = 0$ must have a double root. The coefficients of $F(t)$ are of order eight in a, b, c, d; hence its discriminant V is of order one hundred and twelve.

When R_5 has an inflexion, two stationary planes coincide. Hence IV is a factor of V.

A plane having five point contact with R_5 counts for two stationary planes, that is, the parameter of the point of contact must be a double root of $F(t)$. To find the condition for such a plane, put $t_1 = t_2 = t_3 = t_4 = t_5$ in equations I and I'. The resulting equations are

$$5P_1 t^4 + 10P_2 t^3 + 10P_3 t^2 + 5P_4 t + P_5 = 0.$$
$$q_0 t_5 + 10q_2 t^3 + 10q_3 t^2 + 5q_4 t + q_5 = 0.$$

Their resultant VI is of degree thirty-six in a, b, c, d. It follows that VI is a factor of V.

When R_5 has an actual double point, two different values of the parameter correspond to the double point. Let t and u be these two values and let ρ and σ be the two factors of proportionality. Hence we have

$$\rho x = \sum_{i=0}^{5} a_i t^{5-i};\ \rho y = \sum_{i=0}^{5} b_i t^{5-i};\ \rho z = \sum_{i=0}^{5} c_i t^{5-i},\ \rho w = \sum_{i=0}^{5} d_i t^{5-i};$$

and

$$\sigma x = \sum_{i=0}^{5} a_i u^{5-i};\ \sigma y = \sum_{i=0}^{5} b_i u^{5-i};\ \sigma z = \sum_{i=0}^{5} c_i u^{5-i};\ \sigma w = \sum_{i=0}^{5} d_i u^{5-i}.$$

Multiplying the first equations by σ and the second by ρ and subtracting we get four equations of the form

$$\sum_{i=0}^{5} a_i (\sigma t^{5-i} - \rho u^{5-i}) = 0 \text{ for } a, b, c, d.$$

In order to eliminate t, u, and $\frac{\rho}{\sigma}$, we form the four following identical equations

$$x(\sigma t^i - \rho u^i) + y(\sigma t^{i+1} - \rho u^{i+1}) + z(\sigma t^{i+2} - \rho u^{i+2}) = 0 \qquad (i = 0, 1, 2, 3).$$

where $\frac{x}{z} = tu$, $\frac{y}{z} = -(t + u)$. Take any two of these equations as the first and second and eliminate the expressions $(\sigma t^i - \rho u^i)$, $i = 0, 1, - 5$.

This gives the determinant

$$\begin{vmatrix} a_5, & a_4, & a_3, & a_2, & a_1, & a_0, \\ b_5, & b_4, & b_3, & b_2, & b_1, & b_0, \\ c_5, & c_4, & c_3, & c_2, & c_1, & c_0, \\ d_5, & d_4, & d_3, & d_2, & d_1, & d_0, \\ x, & y, & z, & 0, & 0, & 0, \\ 0, & x, & y, & z, & 0, & 0, \end{vmatrix} = 0.$$

There are six terms in this equation, and five other similar equations in x, y, z. Hence, there are enough equations to eliminate $x^2, xy, y^2, xz, zy, z^2$. Denoting by (i, k) the determinant formed by striking out the $(5-i)$ *th* and $(5-k)$ *th* columns of the constants, we find the condition VII for a double point to be

$$\begin{vmatrix} (01), & (02), & (12), & (03)-(12), & (13), & (23) \\ (02), & (03)+(12), & (13), & (04), & (14)+(23), & (24) \\ (12), & (13), & (23), & (14)-(23), & (24), & (34) \\ (03), & (04)+(13), & (14), & (23)+(05), & (24)+(15), & (25) \\ (13), & (14)+(23), & (24), & (15), & (25)-(34), & (35) \\ (23), & (24), & (34), & (25)-(34), & (35), & (45) \end{vmatrix} = 0.$$

This condition is of degree twenty-four.

To find the condition for a cusp replace one of the equations in x, y, z, by the identity $x + ty + t^2 z = 0$, and eliminate x^2, etc., as before. The roots t and u are given by the equation $t_0 t^2 + t_1 t + t_2 = 0$ where t_0, t_1, t_2 are of degree twenty in a_i, b_i, c_i, d_i. But at a cusp $t = u$. Hence, the condition for a cusp VIII, is of order forty. In order that R_5 may have a cusp, VII and VIII must both be satisfied,* and $F_8(t)$ has a triple root.

The equation determining the parameters of the points on the quadrisecant can be easily found by means of the equations on page 311. Solving for $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ we find the equation determining the parameters to be of the form $Ut^4 - U_1 t^3 + U_2 t^2 - U_3 t + U_4 = 0$, where U and U_4 are of degree sixteen. When a biquadratic equation has a pair of double roots $G = 0$, and $a^2 I - 1\ 2H^2 = 0$†. Hence, denoting these two equations by IX and IX′, we have the result that if R_5 has a bitangent, equations IX and IX′ of degrees forty-eight and sixty-four respectively must be satisfied. The conditions for four coincident roots gives three equations X, X′, X″, of orders thirty-two, thirty-two and forty-eight respectively that must be satisfied when R_5 has a four-point contact tangent. $F_8(t)$ has a four-fold root.

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* This discussion is taken from a paper by A. Brill, "Über die Doppelpunkte von Curven im Raume deren Geschlecht Null ist." *Math. Annalen*, 1871.

† These symbols H, I, G invariants of a binary quartic, are used by Burnside & Panton, Theory of Equations.

§ 5. *Enumeration of Possible Singularities and the Multiple Roots of* $F(t)$.

The non-singular quintic lying on a hyperboloid will be discussed later by another method. We will now consider the various possible singularities on the other rational quintics. Since the curve has only one quadrisecant, there may be one bitangent if conditions IX and IX′ are satisfied, or one four-point contact tangent when conditions X, X′, X″, are satisfied and $F(t)$ has a four-fold root. When R_5 has a linear inflexion conditions IV and V must be satisfied. For each inflexion $F(t)$ has a double root. If a plane has five-point contact with R_5 conditions V and VI must hold. Each such plane requires that $F(t)$ should have a double root. For a cusp, conditions VII and VIII must be satisfied and $F(t)$ has a triple root. There may be an inflexion at a plane having five-point contact when conditions IV and VI must be satisfied and $F(t)$ has a triple root. A cusp at a point whose osculating plane has five-point contact requires conditions VI, VII, and VIII to be satisfied; $F(t)$ has a four-fold root. A cusp at which the tangent has four-point contact with the curve gives rise to a five-fold root of $F(t)$; and a triple point with all three tangents coincident give rise to a six-fold root.

§ 6. *The Quintic with two Inflexions.*

The equation of the quintic with two inflexions may be written in the form,

$$x = t^4(t+a), \quad y = t^3(t+b), \quad z = t+c, \quad w = t(t+d),$$

in which a, b, c, d, are arbitrary.

Equations I and I′ become

$$\Sigma_2 + b\Sigma_1 + ab = 0, \text{ and } cd\Sigma_3 + c\Sigma_4 + \Sigma_5 = 0;$$

the condition that R_5 lies on a hyperboloid is, $d - c = 0$;

$$F(t) = t^2\left[t^4 + \tfrac{2}{3}(b+5c)\,t^3 + \tfrac{1}{6}(ab+15bc+20cd)\,t^2 + \tfrac{2}{3}\,bc\,(a+5d)\,t + abcd\right] = 0.$$

$F(t)$ is of degree six since ∞ was taken as one point of inflexion.

The equation determining the parameters of the points on S_4 is.

$$t^4 + b\,t^3 + (b^2 - ab)\,t^2 + \frac{bcd\,(a-b)}{d-c}\,t + \frac{bcd^2\,(a-b)}{d-c} = 0.$$

In the case of a four-point inflexion this equation is of the form $(t-x)^4=0$. This gives three possible cases.

(*a*) $x=0$, $b=0$. $F(t)=0$ reduces to $t^4(3t^2+10ct+10cd)=0$. The two remaining roots are equal if $5c=6d$. This curve has linear inflexions at ∞ and $-\frac{5}{3}c$, and a four-point contact tangent at 0. It belongs to the linear complex $P_{13}+2P_{42}=0$.

(*b*) $x=-\dfrac{b}{4}$, $8a=5b$, $b=16d$. Hence

$$F(t)=t^2\left(t+\frac{b}{4}\right)^4=0;\ a=\frac{5b}{8},\ b=b,\ c=\frac{b}{10},\ \text{and}\ d=\frac{b}{16}.$$

It belongs to the linear complex $P_{13}+2P_{43}=0$. (*a*) and (*b*) are the same curves; in (*a*), $t=0$ is a four-point inflexion; in (*b*), it is simply a linear inflexion.

(*c*) $x=-\dfrac{b}{4}$, $8a=5b$, $d=0$. Hence $b=0$, $a=0$, and

$$F(t)=t^6+\frac{10}{3}ct^5=0.$$

This curve has for its equations $x=t^5$, $y=t^4$, $z=t+c$, $w=t^2$. Hence, $t=0$ is a cusp at which the tangent has four-point contact with the curve. It is evident geometrically that the curve cannot belong to a linear complex. It was proved by Picard* that the necessary and sufficient condition that a unicursal twisted curve should belong to a linear complex is that the tangent at the point of contact of every stationary plane should have at least three-point contact with the curve, when the point is not a multiple point.

The conditions that the quadrisecant should become a bitangent are $bd=ca$, and $3b-4a=8d$.

If $F_8(t)$ is a square we have one of the following cases: $b=0$, $c=0$, $ab-25cd=0$, $ac-bd=0$.

If $b=0$, $F(t)=3t^6+10ct^5+10cdt^4=0$.

This curve has been already discussed.

If $c=0$, R_5 is composite.

If $ab=25cd$ we have the following cases:

(*d*) $ab-25cd=0$, $2b^2-25bc+45cd+50c^2=0$.
(*e*) $ab-25cd=0$, $b+5c=0$, $a+5d=0$, $d=-5c$.
(*f*) $ab-25cd=0$, $b+5c=0$, $a+5d=0$, $5c=7d$.

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* Applications de la théorie des complexes linéaires à l'étude des surfaces et des courbes gauches. *Annales de l'Ecole Normale*, 1877.

(*d*) The equations of the curve cannot be simplified by means of the relations.

(*e*) The curve does not belong to a linear complex. $F(t) = t^6 - 50c^2 t^4 + 625c^4 t^2 = 0$.

The four points of contact of the stationary planes are $-5c$, 0, $5c$ and ∞, and are therefore harmonic.

(*f*) This curve belongs to the linear complex

$$P_{13} + 2P_{42} = 0. \quad F(t) = t^2\left[t^4 - \frac{50}{7}c^2 t^2 + \frac{625}{49}c^4\right] = 0.$$

Since the curve belongs to a linear complex and has the four double roots $\frac{-5\sqrt{7}}{7}$, 0, $\frac{5\sqrt{7}}{7}$, ∞, it is possible for a rational quintic curve, not lying on a hyperboloid, to have four linear inflexions, and the four points of inflexion are harmonic. Berzolari in *Atti della Reale Accademia dei Lincei*, 1894, p. 330, in his paper, Su la curva gobba razionale del quint'ordine., fails to notice that when the points of contact of the stationary planes coincide in pairs, there may be four inflexions.

If $ac = bd$, there is only one new type of curve

$$(g) \quad 2b^2 - 25bc + 50c^2 + 36bd - 3ab - 60cd = 0.$$

Combining this with $ac = bd$, and $3b - 4a = 8d$, the conditions for a bitangent, we find $a = \frac{b}{2}$, $c = \frac{b}{4}$, $d = \frac{b}{8}$. This curve belongs to the complex $P_{13} + 2P_{42} = 0$.

$$F(t) = t^2\left[t^4 + \frac{3}{2}bt^3 + \frac{13}{16}b^2 t^2 + \frac{3}{16}b^3 t + \frac{b^4}{64}\right] = 0.$$

Hence the quintic curve can have four linear inflexions and one bitangent.

§ 7. *Some possible types of R_5.*

A quintic with three planes having five-point contact cannot have a linear inflexion. These three planes meet in a point and each counts for three osculating planes. But if the curve has one inflexion the class is eight. Hence, this type of R_5 cannot exist.

$F_8(t) = 0$ can have four double roots in the following cases:

(1). Four planes with five-point contact. The equations could be written in the form

$$x = t^5, \; y = (t-1)^5, \; z = (t-k)^5 \; w = 1.$$

(2). Two five-point planes and two inflexions.
(3). One five-point plane and three inflexions.
(4). Four inflexions.
(5). Four inflexions and one bitangent.

The curve with cusps at 0 and ∞ has for its equations, $x = t^4(t+a)$, $y = t^3$, $z = t^2$, $w = t + d$. By applying the analytic test it may be found that it is impossible that this curve should belong to a linear complex. Hence, there is no rational quintic having two cusps and an inflexion.

$$F(t) = 4t^5 + (15d + a)t^4 + 4adt^3 = 0.$$

The curve $x = t^5$, $y = t^3$, $z = t^2$, $w = 1$ has two cusps, the osculating plane at each having five-point contact. It belongs to the complex $P_{14} = 5P_{23}$.

If R_5 has a cusp at a plane having five-point contact, and one inflexion its equations may be written

$$x = t^5,\ y = t^3(t+a),\ z = t^2,\ w = t + b.$$

This curve belongs to the linear complex $P_{14} - 2P_{23} = 0$ if $2a = 5b$. Hence, a R_5 exists having one cusp with osculating plane having five-point contact and two inflexions.

$$F(t) = 6t^6 + (4a + 20b)t^5 + 15abt^4 = 0.$$

Applying the condition that $F(t) = 0$ should have equal roots we find that if $a = 10b$ there is a plane with five-point contact.

The equations of R_5 with a triple point with coincident tangents at 0 and a plane having five-point contact at ∞ may be written

$$x = t^5,\ y = t^4,\ z = t^3,\ w = 1. \quad F(t) = t^6 = 0.$$

This curve cannot belong to a complex. It is impossible to set up the equations of R_5 with a triple point of this kind and a linear inflexion.

§ 8. *The Rational Quintic of the Second Kind.*

The rational quintic of the second species is the partial intersection of a quadric surface F_2 with a ruled surface of the fourth order having a quadrisecant of R_5 for triple line. The rational quintic of the second kind has ∞^1 quadrisecants. All these lines form one system of generators of F_2 the only quadric surface on which the curve lies. The generators forming the other system are simple secants of R_5^0. Our quintic is of type (1.4).

The curve R_5^0 can be depicted on a conic C_2. The plane of C_2 will cut F_2 in another conic C'_2; these conics intersect in four points, through each of which passes one quadrisecant, S_4. The pencil of planes through any one of these four quadrisecants will cut R_5 in one other point, and C_2 once, hence, the points, of R_5 and C_2 are in $(1 . 1)$ correspondence. The four points on a quadrisecant will define a biquadratic involution of R_5. These points will depict into a biquadratic involution on C_2. The lines joining corresponding points of a tetrad will envelop a curve of class three, I_3 and order six. A tetrad on C_2 and any other point correspond to five points lying in a plane. If the involution curve I_3 is supposed given one can easily find the images of the three points which, with the given point t on C_2 form a tetrad by drawing the three tangents from t to I_3. These will meet C_2 in the required points t', t'', t'''.

The images t_1, t_2, t_3 of any three points of R_5 are given; to find the images of the two points t_4 and t_5 which lie in a plane with the three given points. For convenience of notation, we will consider any point t_i on C_2 as the image of (t_i) on R_5. The pencil of planes having for axis the line $(t_1)\,(t_2)$ cuts out a cubic point involution on R_5, to which the triad $(t_3) . (t_4) . (t_5)$ belongs. This will depict into a cubic involution on C_2, in which t_4, t_5 complete the triad corresponding to t_3. The plane of the pencil containing the quadrisecant through (t_1) cuts R_5 in the points (t'_1), (t''_1), (t'''_1) which belong to the cubic involution. Similarly, the plane containing the quadrisecant through (t_2) cuts out the triad (t'_2), (t''_2), (t'''_2). Hence the cubic involution on C_2 is fixed by the two triads t'_1, t''_1, t'''_1; t'_2, t''_2, t'''_2. From t_1 draw the three tangents to I_3, which meet C_2 in the points t'_1, t''_1, t'''_1. Similarly draw the tangents from t_2 to I_3, and find the points t'_2, t''_2, t'''_2. The lines $t'_1\,t''_1$, $t''_1\,t'''_1$, $t'''_1\,t'_1$, $t'_2\,t''_2$, $t''_2\,t'''_2$, $t'''_2\,t'_2$, are tangent to I_3, and also touch $J_2(t_1 t_2)$, the involution curve enveloped by the lines joining corresponding points in the cubic involution. These six tangents determine a Brianchon point 0_{12}. Hence, if t_3 is given draw the two tangents from t_3 to $J_2\,(t_1\,t_2)$; their points of intersection with C_2 are the required points t_4, t_5. The line $t_4\,t_5$ also touches $J_2\,(t_1\,t_2)$.

To determine $t_4\,t_5$, one could use instead of $t_1\,t_2$, the cubic involutions determined by $t_2\,t_3$ or $t_3\,t_1$, whose involution curves are $J_2\,(t_2\,t_3)$ and $J_2\,(t_3\,t_1)$. The tangents drawn from t_3 to either of these conics will meet C_2 in the points t_4, t_5.

If five points t_1, t_2, t_3, t_4, t_5 are the images of five points lying in a plane the cubic involution is determined by any one of the ten pairs of points $t_i\,t_k$. These determine ten corresponding involution conics $J_2\,(t_i\,t_k)$. From the previous case we know that $J_2\,(t_i\,t_j)$ is touched by three tangents $t_k\,t_l$, $t_l\,t_m$, $t_m\,t_k$. Each conic

$J_2(t_i t_j)$ is touched by three lines joining the t's whose subscripts are different from i and j. Each line $t_i t_j$ is tangent to three different conics $J_2(t_p t_q)$, where p and q are different from i and j. Corresponding to the ten conics there are ten Brianchon points 0_{ij}.

We have seen that corresponding to the point t_1, there are three lines $t_1' t_1''$, $t_1'' t_1'''$, $t_1''' t_1'$, which are tangent to I_3 and $J_2(t_1 t_2)$. These lines are also tangent to $J_2(t_1 t_3)$, $J_2(t_1 t_4)$, $J_2(t_1 t_5)$. The scheme may be expressed thus:

$$t_1' t_1'',\ t_1'' t_1''',\ t_1''' t_1', \text{ touch } J_2(t_1 t_2),\ J_2(t_1 t_3),\ J_2(t_1 t_4),\ J_2(t_1 t_5),$$

and similar expressions for the others. Thus we see that of these fifteen lines, tangent to I_3, each set $t_i' t_i''$, $t_i'' t_i'''$, $t_i''' t_i'$, touches four involution conics $J_2(t_i t_k)$ where k has the four values different from i. We can summarize our results as follows:

Five points t_1, t_2, t_3, t_4, t_5 *are the images of five coplanar points of* R_5 *if the three lines joining any three of them in pairs touch the involution conic* $J_2(t_i t_k)$ *determined by the triads on the quadrisecants through the other two.*

Or:

Five points t_1, t_2, t_3, t_4, t_5 *are images of five coplanar points of* R_5 *when the line joining any two of them is tangent to each of three involution conics determined by the three pairs of triads on the quadrisecants through the other three points.*

Or:

Five points t_1, t_2, t_3, t_4, t_5 *are images of five coplanar points of* R_5 *when the three lines joining the points of intersection with* O_2 *of the three tangents to the involution curve* I_3 *from any one of the points* t_i *touch four involution conics* $J_2(t_i t_k)$, $k \neq i$.

If two of the points (t_1) (t_2) coincide, the cutting plane becomes a tangent plane. The line $t_1 t_2$ now becomes a tangent to C_2, and the points $t_1' t_2'$, $t_1'' t_2''$, $t_1''' t_2'''$ respectively coincide. The conic $J_2(t_1 t_2)$ now touches the sides of the triad $t_1' t_1'' t_1'''$ at their points of contact with I_3. Hence, *three points* (t_3) (t_4) (t_5) *lie in a plane with the tangent at* (t_1) *when the lines joining their images all touch the involution conic* $J_2(t_1, t_1)$, *which touches the sides of the triad belonging to* t_1 *at their points of contact with* I_3.

If (t_3) and (t_4) also coincide the tangent at (t_3) cuts the tangent at (t_1), and the plane becomes a double tangent plane. The line joining t_3 to t_5 is tangent to $J_2(t_1 t_1)$, t_5 being the point of contact. Similarly, $t_1 t_5$ touches $J_2(t_3 t_3)$ the point of contact being t_5 also. $J_2(t_1 t_3)$ is touched by $t_1 t_3$, $t_3 t_5$, $t_5 t_1$. The conic $J_2(t_3 t_5)$ is touched by $t_1 t_3$, the point of contact being t_1. Similarly other conics J_2 are touched by certain lines. Since t_3 and t_4 coincide, the tangent at t_3 touches $J_2(t_1 t_1)$. Hence, *to find the image of the point of contact of a tangent which touches* R_5 *in such a point that the tangent at that point cuts the tangent at* $t_1(t_1)$, *construct the conic* $J_2(t_1 t_1)$. *The common tangents to* $J_2(t_1 t_1)$ *and* C_2 *will touch* C_2 *in the required points.* There are four common tangents; therefore the given tangent is cut by four others, hence the *osculating developable is of order eight.*

Since t_3 and t_4 coincide, t_5 becomes the point of contact of the tangent from t_3 to $J_2(t_1 t_1)$. But t_5 must lie on C_2. Hence to find the image t_5 of the residual point of contact of the double tangent plane at (t_1) and (t_3) find the point of intersection t_5 of $J_2(t_1 t_1)$ with C_2 such that $t_5 t_3$ is tangent to $J_2(t_1 t_1)$. The four points of intersection are the images of the four residual points corresponding to the points of contact of the tangents which meet the tangent at (t_1). Hence, *to find the image of the second point of contact of the double tangent plane through* t_1, *construct the involution conic* $J_2(t_1 t_1)$; *any point of intersection of* $J_2(t_1 t_1)$ *with* C_2 *will be the image of* (t_5), *the residual point in the double tangent plane; the intersection of the tangent to* $J_2(t_1 t_1)$ *at* t_5 *with* C_2 *will be the required point* t_3. Since there are four points t_5 we have another proof that every tangent of R_5 is met by four others, or that through each point of the curve R_5 pass four planes, tangent in the same point and in another point of the curve.

If the image of t_3 is given instead of that of t_1 we have seen that $t_5 t_1$ is tangent to $J_2(t_3 t_3)$ at the point t_5. The points t_5 are fixed points on C_2, so that the lines $t_1 t_5$ must be tangent to $J_2(t_3 t_3)$ at the points t_5, and the points t_1 are the points of intersection of these tangents with C_2. Hence, *if* t_1, t_3, *are the images of the two points of contact of a double tangent plane, the four bitangent planes tangent to* R_5 *in* (t_1) *and in another point of the curve have the same residual points as the four bitangent planes tangent to* R_5 *in* t_3 *and in another point of the curve. The involution conics* $J_2(t_1 t_1)$, $J_2(t_3 t_3)$ *intersect* C_2 *in the same four points* t_5; *the lines* $t_5 t_3$, $t_5 t_1$ *are tangent to* $J_2(t_1 t_1)$, $J_2(t_3 t_3)$, *respectively.*

Between the points t_1 and t_3 on the carrier C_2 there is a symmetric (4, 4) correspondence with eight coincidences. At these coincidences the two points of contact of the tangent plane coincide and the plane becomes a stationary tangent plane. Hence R_5 *has eight stationary planes.* The lines $t_1 t_3$ envelop the direction curve D_4 of class four and order twelve. The points of contact of the carrier C_2 with the eight tangents common to D_4 and C_2 are the images of the points of contact of the eight stationary planes of R_5; for in each of these points, t_1 coincides with its corresponding point t_3. The twenty-four branch points in this (4, 4) correspondence are the points of intersection of D_4 with C_2. For two coincident points correspond to each branch point. Hence, two tangents come together in the branch point, or each branch point is on D_4.

If $t_1 = t_2 = t_3$ then the plane becomes an osculating plane to R_5 at (t_1). Hence, *to find the images of the remaining points of intersection of* R_5 *with the osculating plane at* (t_1), *find the conic* $J_2(t_1 t_1)$, *and from* t_1 *draw the two tangents to* $J_2(t_1 t_1)$; *they will cut* C_2 *in the coplanar points* $t_4\, t_5$.

To determine how many osculating planes of R_5 can be drawn through any point of the curve, and osculate the curve elsewhere, suppose the point t_5 given. Lines from t_5 are required such that the lines $t_5 t_1$ touch the conic $J_2(t_1 t_1)$ fixed by the other point of intersection with C_2. Corresponding to each point t_1 is only one conic $J_2(t_1 t_1)$. How many conics $J(t, t)$ are touched by an arbitrary secant $t_5 t_1$? Let the tangent at (t) cut (t_5) (t_1). The cubic involution determined by the pencil of planes through the tangent at (t) will determine the conic $J_2(tt)$. In any plane of the pencil there are three other points (t). The lines joining the images of these points touch $J_2(tt)$ Hence $t_5 t_1$ will touch $J_2(tt)$ if (t_5) (t_1) is in a plane with the tangent at (t). Hence $t_5 t_1$ touches as many conics $J_2(tt)$ as there are tangent lines that cut the chord (t_5) (t_1), that is, eight. There is a (1.8) correspondence between the points (t_1) and the lines (t_5) (t_1). A line (t_5) (t_1) determines one point (t_1) and eight conics $J_2(t, t)$. The point (t_1) determines one conic $J_2(t\,.\,t)$. As t_1 describes C_2 there may be nine coincidences of $J_2(t_1 t_1)$ with one of the eight conics $J_2(tt)$. When $(t_1) = (t_5)$ the plane becomes an osculating plane and only four other tangents cut $(t_5)(t_5)$; that is, three have disappeared, hence $t_1 = t_5$ is a triple self-corresponding element when $t = t_1$. Therefore, *from any point of* R_5 *there can be drawn six planes which osculate the curve elsewhere. The osculating developable is of class nine.*

The biquadratic involution on C_2 defined by the quadrisecants of R_5 has six double points d_1, d_2, d_3, d_4, d_5, d_6, and twelve corresponding branch points. The double points are the points of contact of the six tangents common to I_3 and C_2; the branch points are the points of intersections of the two curves. The six double points are the images of the points of contact of the six quadrisecants which are tangent to R_5. The branch points are the remaining points of intersection of these six quadrisecants with the curve.

The characteristic numbers for the (1 . 4) unicursal R_5 are in the general case:

$$\begin{array}{ll} n = 5. & g = 20. \\ r = 8. & x = 16. \\ m = 9. & y = 12. \\ h = 6. & \alpha = \ 8. \end{array}$$

An exhaustive classification would include all the possible configurations of the nodal curve and the double developable. The maximum genus of x is 3; if it break up into two curves having tangents of R_5 for bisecants of each, both must be unicursal. If one irreducible component has the tangents for trisecants its maximum genus is 1.

The equations of this quintic can be written in terms of the parameters of the points on two quadrisecants, and on two secants from the other system of generators. By means of a suitable transformation two points on one quadrisecant may be taken as 0, and 1, and one point on the other quadrisecant as ∞. Denoting the parameters of the points on the two secants by m and p, we can write the equations.

$$\begin{aligned} x &= t(t-1)(t-n)f_2(t) \\ y &= (t-n)f_3(t). \\ z &= t(t-1)(t-p)\,.\,f_2(t). \\ w &= (t-p)\,.\,f_3(t). \end{aligned}$$

where, x, y, z, w are the four planes passing through the consecutive sides of the skew quadrilateral, forming the tetrahedron of reference. $\frac{x}{z} = \frac{y}{w} = \frac{t-n}{t-p}$. The curve lies on the hyperboloid $xw = yz$.

We will now consider the various types of (1 . 4) rational quintics. Since $h = 6 = \frac{4\,.\,3}{2}$, there can be no cusps or actual double points. The possible line singularities are bitangents, stationary tangents, or tangents with four-point contact.

1. *One Bitangent.*

The biquadratic involution on C_2 is now determined by one group $t_1 t_2 t_3 t_4$ and a group consisting of two double elements $d_{12} d_{34}$. I_3 is touched by the six sides of the complete quadrilateral $t_1 t_2 t_3 t_4$, the line $d_{12} d_{34}$ is double tangent to I_3, its points of contact forming with $d_{12} d_{34}$, a harmonic range. The tangents to C_2 at $d_{12} d_{34}$ touch I_3; and I_3 can be considered as the envelope of the tangents drawn at the intersections of $d_{12}d_{34}$ with the conics circumscribed to the quadrilateral $t_1t_2t_3t_4$. This quadrilateral can be replaced by any other tetrad of the involution. Hence the points of contact of the line $d_{12}d_{34}$ with I_3 are points on the two conics which touch $d_{12}d_{34}$ and are circumscribed to any quadrilateral whose vertices represent a tetrad of the biquadratic involution. Every curve of class three with a double tangent can enter as involution curve for ∞^1 biquadratic involutions. It is only necessary to circumscribe a conic to any quadrilateral circumscribed to I_3. We have, $x = 15$, $y = 11$, $\omega = 1$.

The equations are:

$$\begin{aligned} x &= t^2(t-1)^2(t-n), \\ y &= \quad (t-n)f_3(t), \\ z &= t^2(t-1)^2(t-p), \\ w &= \quad (t-p)f_3(t), \end{aligned}$$

and $xw = yz$ is the equation of the hyperboloid on which R_5 lies.

2. *Two Bitangents.*

The involution on C_2 is now determined by two groups consisting of double elements. Let the elements of these groups be $d_{12}d_{34}$, $d'_{12}d'_{34}$. The involution curve of class three has now two double tangents and therefore, breaks up into a conic J_2, and J_1, the point of intersection of $d_{12}d_{34}$ and $d'_{12}d'_{34}$, the two double tangents. J_2 must touch the two double tangents and the tangents to C_2 at the points d_{12}, d_{34}, d'_{12}, d'_{34}. The points of contact of this conic with these double tangents are harmonic conjugates to J_1 with respect to $d_{12}d_{34}$ and $d'_{12}d'_{34}$, respectively, for the same reasons as in the case for one bitangent. The line I joining these points is therefore the polar of J_1 with respect to J_2 and C_2. Hence I is a side and J_1 the opposite vertex of the self-conjugate triangle common to both conics. If the two tangents are drawn from J_1 to C_2, their points of contact e_{13}, f_{13} represent the two remaining double elements of the involution. Since e_{13} and f_{13} lie on the polar of J_1 we see that they separate harmonically each pair $d_{12}\,d_{34}$, $d'_{12}\,d'_{34}$. Hence, *if a biquadratic involution contains two groups consisting only of double elements, the harmonic element pair common to both these groups represents the fifth and the sixth double element of the involution.*

The Cayley numbers are:

$$x = 14, \quad y = 10, \quad \omega = 2.$$

The equations are obtained from last case by replacing $f_3(t)$ by $(t-k)^2$.

3. *Three Bitangents.*

The biquadratic point involution on C_2 now contains three groups consisting only of double elements. The involution is fixed by two such groups. In the previous case it was shown that the two remaining double elements are the points of contact of the tangents drawn from J_1 to C_2. Let the lines joining $d_{12}\,d_{34}$, $d'_{12}\,d'_{34}$ be D, D' respectively, and the tangents to C_2 at these points D_{12}, D_{34}, D'_{12}, D'_{34}. The lines D, D' can be chosen as conjugate polars of C_2. Then $D_{12}\,D_{34}$ meet on D' and $D'_{12}\,D'_{34}$ meet on D. J_2 breaks up into the two points which lie on the polar of J_1 with respect to C_2 and with J_1 form the vertices of a self-conjugate triangle with respect to C_2. Hence, *if a biquadratic point involution on C_2 possesses three groups consisting only of double elements the involution curve I_3 breaks up into three points which are the vertices of a self-conjugate triangle with respect to C_2; the tetrads form quadrilaterals inscribed to C_2 which have these points for intersections of the diagonals. Or, the ordinary tetrads of the involution consist of the vertices of the ∞^1 quadrangles inscribed in C_2 which have the self-conjugate triangle as diagonal triangle.*

The Cayley numbers for this case are:

$$x = 13, \quad y = 9, \quad \omega = 3.$$

Since there are three double tangents there must be a specialisation in the formulas. Taking 0, 1, and ∞ as three roots the involution can be written in the form

$$t^2(t-1)^2 = m(t-k)^2.$$

If this has a pair of equal roots for one group there must be some values of m and k for which this equation is a perfect square. These give $m = -1$, $k = 1/2$. The parameters of the points of tangency of the two bitangents are 0, 1, and ∞, $1/2$. The equations are obtained from the last case by putting $k = 1/2$. Since the equations determining m and k are linear the curve cannot have more than three bitangents. This is also evident from the fact that each bitangent requires two of the six double elements of the biquadratic involution on C_2; moreover it would be impossible to have an involution curve of class three with four double tangents.

4. *One Linear Inflexion.*

One group of the biquadratic involution on C_2 consists of the points $t_{123}\,t_4$. Three tangents can be drawn to I_3 from t_{123}, hence t_4 is a cusp on the involution curve I_3 and $t_{123}t_4$ is the cuspidal tangent. The three tangents t_1t_2, t_2t_3, t_3t_1, coincide in the tangent to C_2 at t_{123}. Hence I_3 touches C_2 at the point t_{123}. Hence we see that a triple element in the biquadratic involution absorbs two double elements.

The Cayley numbers are:

$$\begin{array}{ll} n=5 & g=15 \\ h=6 & x=15 \\ v=1 & y=12 \\ r=8 & \alpha=6 \\ m=8 & \end{array}$$

The equations are:

$$\begin{aligned} x &= t^3(t-1)(t-n) \\ y &= (t-n)\,.\,f_3(t)\,. \\ z &= t^3(t-1)(t-p)\,. \\ w &= (t-p)\,.\,f_3(t)\,. \end{aligned}$$

5. *One Linear Inflexion and one Bitangent.*

The involution on C_2 is now given by the two groups $d_{12}d_{34}$, $t_{123}t_4$. From the previous treatment we know that I_3 has the line $d_{12}d_{34}$ for double tangent and that its points of contact with I_3 form with $d_{12}d_{34}$ a harmonic range. I_3 has a cusp at t_4 and touches C_2 at t_{123}. It also touches the tangents to C_2 at $d_{12}d_{34}$. This case is possible for I_3 may have both a cusp and a double tangent.

The Cayley numbers become:

$$\begin{array}{ll} n=5. & m=8. \\ h=6. & g=15. \\ v=1. & x=14. \\ \omega=1. & y=11. \\ r=8. & \alpha=6. \end{array}$$

The equations may be written:

$$\begin{aligned} x &= t^3(t-1)(t-n). \\ y &= (t-k)^2(t-n). \\ z &= t^3(t-1)(t-p). \\ w &= (t-k)^2(t-p). \end{aligned}$$

6. *Two Linear Inflexions.*

The involution on C_3 contains two groups $t_{123}t_4$, $t'_{123}t'_4$. The involution curve I_3 has cusps at t_4, t'_4. The lines $t_{123}t_4$, $t'_{123}t'_4$ are cuspidal tangents, and I_3 will touch C_2 at t_{123}, t'_{123}.

$$\begin{array}{ll} n = 5. & g = 11. \\ h = 6. & x = 14. \\ v = 2. & y = 12. \\ r = 8. & \alpha = \ \ 4. \\ m = 7. & \end{array}$$

The equations of this curve are:

$$\begin{aligned} x &= t^3(t-1)\,(t-n). \\ y &= \quad (t-k)\,(t-n). \\ z &= t^3(t-1)\,(t-p). \\ w &= \quad (t-k)\,(t-p). \end{aligned}$$

7. *Two Linear Inflexions and One Bitangent.*

The involution on C_3 has three groups of the form $t_{123}t_4$, $t'_{123}t'_4$, and $d_{12}d_{34}$. I_3 has $d_{12}d_{34}$ for double tangent, and the lines $t_{123}t_4$, $t'_{123}t'_4$ for cuspidal tangents. I_3 touches C_2 at the points $t_{123}t'_{123}$: t_4 and t'_4 are cusps on I_3. Since there is one double tangent the order of I_3 reduces to four, and the number of cusps reduces to three.

The Cayley numbers are:

$$\begin{array}{ll} n = 5. & m = \ \ 7. \\ h = 6. & g = 11. \\ v = 2. & x = 13. \\ \omega = 1. & y = 11. \\ r = 8. & \alpha = \ \ 4. \end{array}$$

Since there are two triple elements the involution can be written $t^3(t-1) = m(t-k)$. If, in addition there is one group consisting of double elements, this must be a square for some value of m and k.

We have $m = k = -1/8$. Hence the equations for the curve become:

$$\begin{aligned} x &= t^3\,(t-1)\,(t-n)\,. \\ y &= (t+1/8)\,(t-n)\,. \\ z &= t^3\,(t-1)\,(t-p)\,. \\ w &= (t+1/8)\,(t-p)\,. \end{aligned}$$

Since this method gave just one value for m and k it follows that the R_5 with two linear inflexions cannot have more than one bitangent. This can also be shown by means of the involution curve.

8. *Three Linear Inflexions.*

The point involution on C_2 now contains three groups $t_{123}\,t_4$, $u_{123}\,u_4$, and $v_{123}\,v_4$. I_3 has three cusps at t_4, u_4, and v_4; the lines $t_{123}\,t_4$, $u_{123}\,u_4$, $v_{123}\,v_4$ are the cuspidal tangents and must therefore meet in a point. I_3 also touches C_2 at t_{123}, u_{123}, v_{123}.

The Cayley numbers become:

$$\begin{array}{ll} n=5. & g=8. \\ h=6. & x=13. \\ v=3. & y=12. \\ r=8. & \alpha=2. \\ m=6. & \end{array}$$

The involution is given by the equation $t^3(t-1) = m\,(t-k)$. When the three roots are equal, $mk - \frac{m}{4} = 0$, and $\frac{m^2}{16} + \frac{mk}{16} = 0$. Therefore $m = -1/4$, $k = 1/4$. The equations may be written:

$$\begin{aligned} x &= t^3(t-1)\,(t-n). \\ y &= (t-1/4)\,(t-n). \\ z &= t^3(t-1)\,(t-p). \\ w &= (t-1/4)\,(t-p). \end{aligned}$$

The involution has three triple elements for which $t=0$, $t=\infty$, and $t=1/2$. This also shows that the involution cannot have more than three triple elements. Hence, the R_5 cannot have more than three linear inflexions. This is also evident geometrically. For C_2 and I_3 in the case of three linear inflexions touch in three points and three cusps of I_3 are on C_2. Hence the curves meet in twelve points. An additional bitangent or inflexion would require additional points of intersection of C_2 and I_3, which is impossible.

9. *One Tangent of Four-point Contact.*

If the points $t_{12}\,t_{34}$ in the case when one group is composed of double elements become coincident we have one group consisting of a fourfold element. The tangent to C_2 at t_{1234} is a double tangent of I_3 and t_{1234} is one of the points of contact. The other point of contact of the double tangent is the point where it touches the second conic circumscribed to any quadrilateral $t_1 t_2 t_3 t_4$ and tangent to the double tangent. The tangent at t_{1234} counts for three common tangents to C_2 and I_3. Hence, a four-fold point on R_5 absorbs only three of the double elements of the involution.

The number of stationary planes on R_5 reduces to four, and the order of the nodal curve to thirteen; hence, for this case we have $v = 2$, $\omega = 1$, $n = 5$, $h = 6$, $r = 8$, $x = 13$, $m = 7$, $g = 11$, $\alpha = 4$, $y = 11$.

The equations are:

$$\begin{array}{ll} x = t^4 (t - n) . & y = (t - n) \cdot f_3 (t) . \\ z = t^4 (t - p) . & w = (t - p) \cdot f_3 (t) . \end{array}$$

10. *One Four-point Contact Tangent and one Bitangent.*

In the biquadratic point involution considered on p. 330, the points $d_{12} d_{34}$ coincide. The involution curve consists of the point J_1 and the conic J_2. J_2 touches C_2 at the point d_{1234}; it also touches the tangents to C_2 at the points $d'_{12} d'_{34}$, and the line $d'_{12} d'_{34}$.

The Cayley numbers are:

$$\begin{array}{ll} n = 5 . & m = 7 . \\ h = 6 . & g = 11 . \\ v = 2 . & x = 12 . \\ \omega = 2 . & y = 10 . \\ r = 8 . & \alpha = 4 . \end{array}$$

The equations are:

$$\begin{array}{ll} x = t^4 (t - n) . & y = (t - k)^2 (t - n) . \\ z = t^4 (t - p) . & w = (t - k)^2 (t - p) . \end{array}$$

11. *One Four-point Contact Tangent and one Linear Inflexion.*

The point involution on C_2 has two groups d_{1234} and $t_{123} t_4$. The tangent to C_2 at d_{1234} is a double tangent to I_3, and I_3 touches C_2 at d_{1234}. I_3 has a cusp at t_4; $t_{123} t_4$ is the cuspidal tangent; and C_2 touches I_3 at t_{123}.

The Cayley numbers are:

$$\begin{array}{ll} n = 5 . & m = 6 . \\ h = 6 . & g = 8 . \\ v = 3 . & x = 12 . \\ \omega = 1 . & y = 11 . \\ r = 8 . & \alpha = 2 . \end{array}$$

The equations are:

$$x = t^4 (t - n)\,. \qquad y = (t - k)(t - n)\,.$$
$$z = t^4 (t - p)\,. \qquad w = (t - k)(t - p)\,.$$

12. *Two Four-point Contact Tangents.*

The involution on C_2 contains the groups $d_{1234}\, d'_{1234}$. The tangents to C_2 at these points meet in the point J_1. J_2 will touch C_2 at the points $d_{1234}\, d'_{1234}$.

The Cayley numbers are:

$$\begin{array}{lll} n = 5\,. & r = 8\,. & y = 10\,. \\ h = 6\,. & m = 5\,. & \alpha = 0\,. \\ v = 4\,. & g = 6\,. & \\ \omega = 2\,. & x = 10\,. & \end{array}$$

The equations are:

$$x = t^5 \qquad y = t$$
$$z = t^4 \qquad w = 1\,.$$

Two bitangents and one linear inflexion is an impossible case. Consider the involution on C_2 determined by two groups consisting of double elements. It has been shown that I_8 breaks up into a point J_1 and a conic J_2. A triple element t_{123} requires a cusp on I_8 at the point t_4, which is impossible in this case. This can also be shown analytically. The involution determined by the two bitangents is of the form

$$t^2 (t - 1)^2 = m (t - k)^2.$$

If the curve has in addition one linear inflexion this equation must have a triple root for some suitable values of m and k. But this requires conditions which are incompatible and the additional inflexion is impossible.

§ 9. *Curves that belong to a Linear Complex.*

(1) Four linear inflexions.

$$x = t^4 \left(t - \frac{25}{7} c\right), \quad y = t^3 (t - 5c), \quad z = t + c, \quad w = t\left(t + \frac{5c}{7}\right).$$

This curve belongs to the complex $P_{13} + 2P_{42} = 0$.

(2) One four-point inflexion, two linear inflexions. It belongs to the complex $P_{13} + 2P_{42} = 0$.

$$x = t^5, \quad y = t^4, \quad z = t + c, \quad w = t\left(t + \frac{5c}{6}\right).$$

(3) Two cusps at planes with five-point contact.

$$x = t^5, \quad y = t^3, \quad z = t^2, \quad w = 1.$$

This is a (2, 3) curve on the hyperboloid $xw = yz$, and belongs to the complex $P_{14} - 5P_{42} = 0$. The developable is of order six and the nodal curve is a rational R_6 having two double points.

(4) Two four-point inflexions.

$$x = t^5, \quad y = t^4, \quad z = t, \quad w = 1.$$

This is a (1, 4) curve on the hyperboloid $xw = yz$. It belongs to the complex $3p_{14} - 5p_{23} = 0$.

(5). Two linear inflexions and one cusp at a plane with five-point contact.

$$x = t^5, \quad y = t^3(t + a), \quad z = t^2, \quad w = t + \frac{2a}{5}.$$

This quintic belongs to the complex $P_{14} - 2P_{23} = 0$.

(6) Four inflexions, one bitangent.

$$x = t^4\left(t + \frac{b}{2}\right), \quad y = t^3(t + b), \quad z = t + \frac{b}{4}, \quad w = t^4\left(t + \frac{b}{8}\right).$$

The curve belongs to the complex $P_{13} + 2P_{42} = 0$.

A rational quintic curve belonging to a linear complex projects from an arbitrary point in space into a rational plane quintic curve C_5 having five inflexions on a straight line. Dually, the section of the developable is a plane curve with five cusps, and the cuspidal tangents all meet in one point, the pole of the given plane.

CHAPTER II.

ON THE QUINTIC CURVES OF GENUS ONE.

The quintic curve of genus one R_5' projects from one of its own points in a plane quartic curve of genus one, with two double points. Hence, through each point of R_5 pass two trisecants. Any trisecant T is met by three trisecants. The order of the ruled surface formed by the trisecants is two greater. Hence, the trisecant surface F_5 of R_5 is of order five. R_5 is the only double curve of the surface. Hence, F_5 is of genus one.

Every surface of the third order through fourteen points of R_5 cuts R_5 in one and the same fifteenth point. For, pass the pencil of the surfaces through the given fourteen points and four points not on R_5. Each surface cuts R_5 in one more point. But a twisted curve of genus one cannot have a pencil of surfaces that cut it in one variable point besides the fixed points. On the contrary there is an infinite number of adjoined surfaces that cut it in variable point pairs. Hence, the surfaces through fourteen points pass through a fixed fifteenth point. All the F_3 through fourteen points do not contain R_5; for, take five points in the plane α, five points in β, and four in γ. The planes α, β, γ represent a F_3 on which R_5 does not lie.

Take any fourteen points as basis points. Pass F'_3 through these points. F_3 and F'_3 intersect in fifteen points on R_5. Hence R_5 meets F_3 in sixteen points, and R_5 lies on F_3. Hence, if a surface F_3 contains the curve R_5, this counts for fifteen simple conditions.*

A surface of the third order is determined by nineteen conditions. Hence, through R_5 and any four points of space not on R_5, a surface F_3 containing R_5 is determined. All the F_3 through R_5 and three points form a system of ∞^1 surfaces. The surfaces F_3 passing through R_5 form a linear system of ∞^4 surfaces. A line counts for four conditions. Hence through R_5 and any line g of space can be drawn one cubic surface.

Let F_3 be any cubic surface through R_5. The trisecant surface F_5 meets F_3 in a curve of order fifteen, in which R_5 is a double curve. Let A be any point on the residual curve. The trisecant a_1 through A meets R_5 and therefore F_3 in three other points. Hence a_1 is a line on F_3. Hence, the residual curve of order five is composed of five skew lines, a_1, a_2, a_3, a_4, a_5.

Let g and g' be the two transversals of the lines a_1, a_2, a_3, a_4; they meet F_5 in five points each, and are lines on the surface F_3. The fifth point of intersection of each line with F_5 must therefore be on the curve of intersection of F_3 and F_5. Hence, both lines g and g' cut a_5, and every quintuple a_1, a_2, a_3, a_4, a_5, is a quintuple with two transversals.

Pass an F_3 through g and R_5. g meets F_5 in five points through which pass five trisecants a_1, a_2, a_3, a_4, a_5, each of which has four points on F_3 and therefore lies on F_3. The intersection of F_3 and F_5 consists of R_5 counted twice and a_1, a_2, a_3, a_4, a_5, which have two transversals g and g'. Since g' can be found by means of g, and g by means of g', it follows that the lines of space are paired such that each pair of lines is cut by the same five trisecants, and if four trisecants a_1, a_2, a_3, a_4 are met by the two transversals g and g', then g and g' cut the same fifth trisecant a_5 and are corresponding lines. The same result will now be obtained by depicting F_3 and R_5 on a plane.

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* See also Halphen, Sur la classification des courbes gauches algébriques, *Journal de l'Ecole polytechnique.* Vol. 52, p. 41.

Take a line a_1 and five other lines b_2, b_3, b_4, b_5, b_6 which do not intersect cutting it. These lines are on a cubic surface. For take four points in which a_1 cuts say $b_2 \dots b_5$, and three other points on each of the lines $b_2 \dots b_6$. The nineteen points determine a cubic surface through these six lines. Four skew lines in space have in general two transversals, hence through b_3, b_4, b_5, b_6 we can draw another transversal a_2 meeting F_4 in four points, and hence lying on the surface. Similarly we can find the lines a_3, a_4, a_5, a_6. Draw a line cutting the lines a_2, a_3, a_4, a_5; it will also cut a_6. Call the line b_1. These lines a_i, b_i form a *double six.* The remaining fifteen lines lie in the planes determined by a_i and b_k $(i \neq k)$. The lines $a_i b_k$ and $a_k b_i$ determine the same line r_{ik}. It can now be seen that any two b_i cut four $a_k (k \neq i)$, *e. g.*, b_2 and b_3 cut a_1, a_4, a_5, a_6; b_1 and b_6 cut a_2, a_3, a_4, a_5. b_1 also cuts r_{12}, r_{13}, r_{14}, r_{15}, r_{16}; b_6 cuts r_{61}, r_{62}, r_{63}, r_{64}, r_{65}. Hence a_2, a_3, a_4, a_5, and r_{16} cut both b_1 and b_6.*

Now depict the surface on a plane by means of the lines b_1 and b_6. The lines a_2, a_3, a_4, a_5, r_{16} meet both b_1 and b_6 and hence project into points. The other five lines meeting b_1 are a_6 and r_{12}, r_{13}, r_{14}, r_{15}; these project into the lines of a pencil through b_1. Similarly the lines a_1, r_{62}, r_{63}, r_{64}, r_{65} project into the lines of a pencil through b_6. Let b_1 and b_6 cut the plane in the points b_1 and b_6. a_1 and a_6 intersect in r_{16}; a_2 is cut by r_{12} and r_{62}, hence a_2 projects into the intersection of r_{12} and r_{62}; similarly for a_3, a_4, a_5. There are ten other lines r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45}, b_2, b_3, b_4, b_5 which do not cut either b_1 or b_6. They depict into conics. The line r_{23} cuts a_2, a_3, b_2, b_3, r_{14}, r_{56}, r_{15}, r_{46}, r_{16}, r_{45}; its image is a conic through the five points a_2, a_3, r_{16}, b_1, b_6. The line b_2 cuts $a_i (i \neq 2)$ and $r_{i2} (i \neq 2)$; hence it depicts into a conic through the points a_3, a_4, a_5, b_1, b_6. The complete depiction of the twenty-seven lines consists of:

(1) two points b_1, b_6.

(2) five points a_2, a_3, a_4, a_5, r_{16}.

(3) five lines a_6, r_{12}, r_{13}, r_{14}, r_{15} joining b_1 to the points r_{16}, a_2, ...

(4) five lines a_1, r_{62}, r_{63}, r_{64}, r_{65} joining b_6 to the points r_{16}, a_2, ...

(5) ten conics r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45}, b_2, b_3, b_4, b_5, passing through b_1, b_6, and three of the points $a_2 \dots r_{16}$.

Suppose a $R_5 (p = 1)$ passed through three points on a_2, a_3, a_4, a_5, r_{16}. From any point of R_5 one line can be drawn cutting b_1 and b_6. These lines generate a ruled surface of order ten on which b_1 and b_6 are five-fold lines, and the trisecants a_2, a_3, a_4, a_5, r_{16} are triple generators; there are no other nodal lines. The section of the surface by the plane of depiction is the image of R_5. This is a plane curve C_{10} of order ten having five-fold points at b_1 and b_6 and triple points at a_2, a_3, a_4, a_5, r_{16} and no other double points. Its genus is $P = \frac{9\,.\,8}{2} - 2\,.\,\frac{5\,.\,4}{2} - 5\,.\,\frac{3\,.\,2}{2} = 1$.

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* Salmon, Analytical Geometry of Three Dimensions.

The plane of depiction cuts F_3 in a cubic curve C_3 which passes simply through the seven fundamental points, and cuts every other a, b, r in one point apart from a fundamental point.

Any line, say a_6, that meets b_1, but not b_6, goes into a line cutting C_{10} in ten points, of which five are at b_1, three at r_{16}, and two others are simple points. Hence, a_6, and similarly r_{12}, r_{13}, r_{14}, r_{15}, a_1, r_{62}, r_{63}, r_{64}, r_{65} are bisecants of the original curve R_5. The line r_{23}, not cutting b_1 or b_6 goes into a conic C_2 through b_1, b_6, a_2, a_3, r_{16}, cutting C_{10} in twenty points. There is just one point of intersection apart from the fundamental points. Hence r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45}, b_2, b_3, b_4, and b_5 are unisecants of R_5. The lines of F_3 meet R_5 as follows:

(1) a_2, a_3, a_4, a_5, r_{16}, are trisecants.
(2) a_6, r_{12}, r_{13}, r_{14}, r_{15}, a_1, r_{62}, r_{63}, r_{64}, r_{65} are bisecants.
(3) r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45}, b_2, b_3, b_4, b_5, are unisecants.*
(4) b_1, b_6, do not meet R_5.

From this it follows that the lines of space are in pairs a, a', for the surface through fifteen points on R_5 and four points on b_1 also contains b_6, and similarly the surface through R_5 and b_6 contains b_1. Hence every line of space belongs to such a pair and with R_5 determines a surface F_3.

Let g go through a point A of R_5 in which the trisecants a_1, a_2 meet. Then the three trisecants a_3, a_4, a_5 through the three further points of intersection of g and F_5 form with a_1 and a_2 a quintuple. The line of intersection g' of the planes containing a_1, a_2, and a_3, a_4, meets four lines of the quintuple and therefore corresponds to g. Hence; to the lines g of the bundle with vertex at a point A on R_5 correspond the lines of the plane α containing the two trisecants a_1, a_2 through the point A.

If g lies in the plane α, g', coincides with g; hence, the lines of the plane

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*D. Montesano in "Su la curva gobba di 5° ordine e di genere 1"; *Rendiconte dell 'Accademia di Napoli delle scienze fisische e matematiche*, 2nd series, vol. II, 1887, p. 182, states incorrectly that of the twenty-seven lines of F_3, twenty are bisecants.

pencil with vertex at a point A on R_5, and lying in the plane α of the trisecants a, a' through A are self-corresponding. From this, it follows that every trisecant is paired with itself.

Next, suppose g to cut R_5 in the points A, B, through which pass the trisecants a_1, a_2, a_3, a_4. The plane α containing a_1, a_2, and the plane β containing a_3, a_4 intersect in the line g' corresponding to g. Hence, to the line joining two points A, B of R_5 corresponds the line of intersection of the planes α, β containing the trisecants at the two points. Since g and g' are both met by the same fifth trisecant of the quintuple, it follows that the lines AB and $(\alpha\beta)$ are met by the same trisecant.

The surface F_3 containing a pair of corresponding lines a, a', and a surface F_3' containing a pair b, b' intersect in R_5' and a curve R_4^0 of order four and genus zero. Since a and a' meet F_3' in three points they are trisecants of R_4; similarly, it follows that b, and b' are also trisecants. The lines a, a', b, b', therefore belong to a fixed hyperboloid, since R_4^0 is a (1, 3) curve on one hyperboloid.

Hence, by the curve R_5' is determined a linear complex so that any point of R_5 corresponds to the plane determined by the trisecants through that point, and the line joining any two points corresponds to the intersection of the two corresponding planes; since the trisecants are self-corresponding lines in the system, they belong to a linear complex.

Let a pair of conjugate lines a, a' approach coincidence and let F_3 be the surface of the system containing a. Then a is a line of the complex. Each plane through a has two points of contact on a; hence, there is a (1, 2) correspondence between the planes through a and the points of a. Each plane through a has its pole on a; hence, there is a (1.1) relation between the planes through a and their poles on a. Two such correspondences can have three common points. But a is met by five trisecants, and therefore there are five planes such that the pole is at one point of contact. Hence, one point of contact is fixed, and the planes through a are projective to the variable point of contact. Therefore, there is a double point of F_3 on a.

Conversely, every point of space is a double point on an F_3 through R_5; there are ∞^3 such surfaces. The tangent cone at the double point P of F_3, meets F_3 in six lines. To study the lines on F_3 depict the surface on a plane α. Each line through P meets F_3 in one more point. The six lines through P depict into points a, b, c, d, e, f, The other fifteen lines of F_3 go into the lines joining the fundamental points. It follows that one line a does not meet R_5, that b, c, d, e, f are bisecants, that the five lines meeting a are trisecants, and that the other ten lines are unisecants. R_5 goes into a plane quintic curve with double points at b, c, d, e, f. Hence this method of depiction gives the simplest monoidal representation of R_5'. The two lines a, a' of the general case coincide in a; it follows that a is a line of the complex.

From what has just been proved it follows that the lines a of the complex and the points P of space are in $(1 . 1)$ correspondence, such that a line a of the complex and R_5 determines a F_3 with a point on a for double point, and that the point P and R_5 determines a F_3 with double point at P, and containing a. Call this correspondence χ.

Every point of R_5 is on all the surfaces F_3; hence, in χ to each point of R_5 correspond all the lines of the complex through that point. The points of R_5 are therefore singular points in χ.

Consider the planes through a trisecant S_3. The poles of these planes are the points of S_3, so that the trisecant corresponds to all its points. Hence, the lines of the trisecant surface F_5 are singular lines in the correspondence χ.

Take any plane α and its pole P in the linear complex. The locus of the points that correspond in χ to the lines of a plane pencil in α with vertex at P is a conic C_2 through P, since there is one corresponding point on each line. The plane α cuts R_5 in five points A, B, C, D, E. To the lines PA, PB, PC, PD, PE correspond the points A, B, C, D, E; hence, C_2 is the conic through A, B, C, D, E. But the polar planes of the points A, B, C, D, E all go through the pole P of α. Hence; *if any plane* α *cuts* R_5 *in the five points* A, B, C, D, E, *the five planes determined by the pair of trisecants through each point all pass through* P *the pole of the plane* α *in the complex, and* P *lies on the conic determined by* A, B, C, D, E.

It has been proved by Fiedler that all self-dual curves of order five are of genus zero.*

CHAPTER III.

ON THE QUINTIC CURVES OF GENUS TWO.

The quintic curve of genus two is the partial intersection of a quadric surface F_2 with a cubic surface, the residual intersection being a trisecant of R_5. We will study it as a $(2, 3)$ curve on a hyperboloid F_2.

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*See W. Fiedler, *Vierteljahresschrift der naturforschenden Gesellschaft in Zurich*, vol. XX, pp. 173–176.

To depict the surface on a plane E take any point A on F_2 but not on R_5.* Any line through A meets F_2 in one point, and the plane of depiction in one point. This gives a (1, 1) correspondence between the points of F_2 and E. The two generators through A meet E in the points O and O', hence O corresponds to all the points on OA, and O' to all the points on $O'A$. The line l joining OO' corresponds to A on F_2; the different points on l correspond to different directions through A. The fundamental elements of the correspondence are O, O' and l on E, and OA, $O'A$, and A on F_2.

It is shown by Clebsch that C_5, the plane projection of R_5 from A has for its equation.

$$\sum C_{abc}\, x^a y^b z^c = 0,$$

subject to the conditions

$$a + b \geqq 2, \text{ and } a + c \geqq 3.$$

Hence, the equation of C_5 is

$$b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^4z + b_5x^3yz + b_6x^2y^2z + b_7x^3z^2 + b_8x^2yz^2 + b_9xy^2z^2 + b_{10}x^2z^3 + b_{11}xyz^3 + b_{12}y^2z^3 = 0.$$

From O eight tangents can be drawn to touch C_5 elsewhere; hence, *eight generators of the second system are tangent two R_5 and cut R_5 in another point.*

Six tangents can be drawn from O' to touch C_5; hence, *six generators of the first system are tangent to R_5.*

The curve C_5 with a double point at O and a triple point at O' is determined by eleven other points. Hence, R_5 is determined by eleven arbitrary points on F_2.

The equation of C_5 in non-homogeneous coordinates is

$$b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^4 + b_5x^3y + b_6x^2y^2 + b_7x^3 + b_8x^2y + b_9xy^2 + b_{10}x^2 + b_{11}xy + b_{12}y^2 = 0.$$

The abscissas of the points in which the line $y = mx$ cuts C_5 are given by the equation

$$(b_1 + b_2m + b_3m^2)\,x^3 + (b_4 + b_5m + b_6m^2)\,x^2 + (b_7 + b_8m + b_9m^2)\,x + (b_{10} + b_{11}m + b_{12}m^2) = 0.$$

The conditions that the three roots of this equation are equal are

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3}.\dagger$$

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* See Clebsch, Vorlesungen über Geometrie, vol. 2, p. 414.

† Burnside & Panton, Theory of Equations, vol. I, p. 84.

Since a_0, a_1, a_2, a_3 are all of order two in m, there cannot be more than four values of m for which all the roots are equal. One value gives the triple point at infinity. Hence, the quintic curve of genus two cannot have more than three inflexional tangents.

Any plane cuts F_2 in a conic C_2 whose image is another conic through O, and O', meeting C_5 in five other points. Hence, the conics through O and O' meet C_5 in five points which are the images of five coplanar points of R_5. The conic may break up into a line through O and a line through O'.

To the five points P_4 in which a line l through any point P in the plane E meets C_5, correspond the five points in which the plane determined by l and A meets R_5. To the points on the planes of the pencil with axis AP correspond the points on the lines of the plane pencil P. If the point P is on C_5 there are only four variable points of intersection.

The projection of the tangent plane at A is OO'.

The projection of the section of R_5 by a tangent plane to F_2 is the points on a line through O and a line through O'.

Draw a pencil of planes with OA for axis. The points in which any plane of the pencil meets R_5 go into O and the three points in which the line of intersection with E cuts C_5. Similarly, the points of R_5 on the planes through $O'A$ lie on the lines of intersection of the planes with E.

All sections by planes that do no pass through A, depict into conics through O and O'.

The ten points of intersection of R_5 with any F_2 lie on ∞^1 F_2, and they are all cut out by a (2.2) curve on the same ruled surface. To these ten points correspond the intersections of C_5 with a C_4 with double points at O and O'.

To a doubly counted plane corresponds a C_4 touching C_5 in five points not in O and O'.

If one draws through these five points and O and O' all the C_4 with double points at O and O' these meet C_5 in five other points, the images of five coplanar points.

Attraction of the Homogeneous Spherical Segment.[*]

BY G. W. HILL.

SECTION I.—*Reduction of the Attraction to Single Quadratures.*

We may suppose the segment treated does not exceed the hemisphere. The centre of the sphere may be adopted as the origin of a system of rectangular coordinates. The straight line going from the centre to the attracted point may be taken as the axis of x; in case this line is indeterminate, the axis of the segment may be taken. The axis of y is put in the plane which contains the axes of x and of the segment; its positive direction is on the side where lies the centre of gravity of the segment. The positive direction of the axis of z is immaterial. In the treatment of the component of the attraction in the direction of x it is convenient to substitute for y and z the variables r and ω such that

$$y = r\cos\omega, \quad z = r\sin\omega.$$

All radicals occuring in the following expressions are to be taken positively.

Denoting the density by ρ, and, for brevity, putting

$$b = x - x',$$

the components of the attraction, severally in the directions of the axes of x and y, are

$$X = \rho\int\int\int\frac{b}{[b^2+r^2]^{\frac{3}{2}}}\,rdr\,d\omega\,dx, \quad Y = \rho\int\int\int\frac{y}{[b^2+y^2+z^2]^{\frac{3}{2}}}\,dy\,dz\,dx.$$

There is no need of mentioning Z as it plainly vanishes. The integrations are understood to be taken in the order of the written differentials.

The equation of the sphere to which the segment belongs is

$$x^2 + y^2 + z^2 = x^2 + r^2 = a^2,$$

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[*] A possible application in geodesy induced the composition of this memoir. Although the simple case where the attracted point lies on the axis of the segment has been several times treated, after a long search, no general investigation could be found. Fearing the matter might have escaped me, I consulted the bibliographical knowledge of mathematical friends, but with only a negative result.

and that of the plane, part of which forms the base of the segment, may be written

$$x\cos\phi + y\sin\phi = a\cos\gamma,$$

where γ is the angular radius of the segment, always in the first quadrant, and ϕ is the angle between the axes of the segment and of x, always less than two right angles. The triple integrals must then be extended so as to cover all points satisfying the two inequations

$$x^2 + y^2 + z^2 < a^2, \qquad x\cos\phi + y\sin\phi > a\cos\gamma.$$

The indefinite integrals of the expressions under the signs of integration in X and Y, the first with reference to r, the second with reference to y, are severally

$$-\frac{b}{\sqrt{b^2 + r^2}} \text{ and } -\frac{1}{\sqrt{b^2 + y^2 + z^2}}.$$

The section of the segment by the plane perpendicular to the the axis of x is discontinuous at the two points

$$x = a\cos(\phi + \gamma) = \mathrm{b}, \qquad x = a\cos(\phi - \gamma) = \mathrm{a}.$$

Therefore, in the derivation of X, we are forced to divide the integration with respect to x into two parts, and to establish three distinct cases. Let us, for brevity, put

$$c = \sqrt{a^2 - x^2}, \qquad d\sin\phi = a\cos\gamma - x\cos\phi.$$

In the first part, the integration is from $\omega = 0$ to $\omega = 2\pi$, and, in the second part, from $r = \frac{d}{\cos\omega}$ to $r = c$. We therefore put

$$A = 2\pi\rho \int\int \frac{b}{[b^2 + r^2]^{\frac{3}{2}}} r\,dr\,dx, \qquad B = \rho \int\int \left[\frac{b\cos\omega}{\sqrt{d^2 + b^2\cos^2\omega}} - \frac{b}{\sqrt{b^2 + c^2}}\right] d\omega\,dx.$$

The statement of the three cases is as follows:

Case I.—Where $\phi + \gamma > 180°$.

$$X = A + B.$$

Case II.—Where $\phi - \gamma$ and $\phi + \gamma$ both lie between 0° and 180°.

$$X = B.$$

Case III.—Where $\phi - \gamma$ is a negative angle.

$$X = A + B.$$

Case III differs from Case I in that the limits of integration for A are different.

In Y the integration with respect to y is from $y = d$ to $y = \sqrt{c^2 - z^2}$, and, in every case

$$Y = \rho \int\int \left[\frac{1}{\sqrt{b^2 + d^2 + z^2}} - \frac{1}{\sqrt{b^2 + c^2}} \right] dz\, dx.$$

Proceeding to remove another sign of integration from these expressions, we note the indefinite integrals of the quantities under the signs of integration in X are severally

$$-\frac{b}{\sqrt{b^2 + r^2}} \text{ and } \operatorname{arc}\sin \frac{b \sin \omega}{\sqrt{b^2 + d^2}} - \frac{b}{\sqrt{b^2 + c^2}}\omega.$$

The first must be taken between the limits $r = 0$ and $r = c$, the second from $\omega = -\operatorname{arc}\sin \frac{\sqrt{c^2 - d^2}}{c}$ to $\omega = \operatorname{arc}\sin \frac{\sqrt{c^2 - d^2}}{c}$. Then, altering the mode of expression of A and B, we have

$$A = 2\pi\rho \int \left[\frac{b}{\sqrt{b^2}} - \frac{b}{\sqrt{b^2 + c^2}} \right] dx,$$

$$B = 2\rho \int \left[\operatorname{arc}\sin \left(\frac{b}{c} \frac{\sqrt{c^2 - d^2}}{\sqrt{b^2 + d^2}} \right) - \frac{b}{\sqrt{b^2 + c^2}} \operatorname{arc}\sin \frac{\sqrt{c^2 - d^2}}{c} \right] dx,$$

and the three cases are, as before,

CASE I.— $X = A + B$. CASE II.— $X = B$. CASE III.— $X = A + B$.

In the expression for A, it has been preferred to write $\frac{b}{\sqrt{b^2}}$ instead of ± 1, as the latter notation involves the necessity of explaining the ambiguous sign; and the arcs in B are supposed to be taken between the limits $\pm \frac{\pi}{2}$.

The integration in Y with respect to z is from $z = -\sqrt{c^2 - d^2}$ to $z = \sqrt{c^2 - d^2}$, and thus

$$Y = \rho \int \left[\log \frac{\sqrt{b^2 + c^2} + \sqrt{c^2 - d^2}}{\sqrt{b^2 + c^2} - \sqrt{c^2 - d^2}} - 2\frac{\sqrt{c^2 - d^2}}{\sqrt{b^2 + c^2}} \right] dx.$$

The components of the attraction are now reduced to single quadratures. The limits of integration for B and Y are from $x = \mathrm{b}$ to $x = \mathrm{a}$; for the A of Case I, from $x = -a$ to $x = \mathrm{b}$: for the A of Case III, from $x = \mathrm{a}$ to $x = a$.

The integration indicated in the expression A is easily accomplished; for Case I we have

$$A = 2\pi\rho \left[\sqrt{(\mathrm{b} - x')^2} - \frac{x'^2 + a^2 - x'a}{3x'^2}(a + x') - \frac{2x'^2 - a^2 - x'\mathrm{b}}{3x'^2} \sqrt{x'^2 + a^2 - 2x'\mathrm{b}} \right];$$

and for Case III we have

$$A = 2\pi\rho\left[-\sqrt{(\mathrm{a}-x')^2}+\frac{x'^2+a^2+x'a}{3x'^2}\sqrt{(a-x')^2}+\frac{2x'^2-a^2-x'a}{3x'^2}\sqrt{x'^2+a^2-2x'\mathrm{a}}\right].$$

SECTION II.—*Reduction of the Attractions to the Canonical Forms for Elliptic Integrals.*

The definite integrals, just obtained as the values of the functions B and Y, have the great advantage of having no discontinuities for the case when the attracted point is on the edge of the segment, or when $x'=a$, $\phi=\gamma$. Still, if we would learn the essential qualities of these functions, the arcs must be made to disappear from under the signs of integration, and their places taken by purely algebraic expressions. For accomplishing this, a ready method is that called integration by parts, the arcs momentarily being regarded as constant. Thus discontinuities are introduced into the values of the definite integrals when the elements of the problem show that the attracted point is at the edge of the segment. However, as we know the components of the attraction are finite in this case, the sum of the discontinuities must vanish. Thus is suggested a method of escaping their consideration. By neglecting every discontinuity as it presents itself, we must necessarily arrive at the correct value of the component. But this course of proceeding does not remove all the difficulties. The general expressions for the components involve terms formed by the product of two factors, of which the first is a multiplier of the second a definite integral. In case the attracted point is at the edge of the segment, the factor vanishes, while the definite integral becomes infinite. It is not possible to remove the factor within the sign of integration as it is variable. The only course to be pursued is to put the factor in the form $A\varepsilon$, and the value of the definite integral in the form $\frac{B}{\varepsilon}$, where ε is an infinitesimal expressing the distance of the elements of the problem from the singular point; then the value of the term is evidently AB. But the determination of A and B is sometimes baffling, especially when, for the purpose of removing imaginary parameters, definite integrals, whose values go at a bound from 0 to ∞, are introduced.

Attending to B and regarding only the first term, after integration by parts, we have the term

$$\left[2\rho x \text{ arc sin}\left(\frac{b}{c}\,\frac{\sqrt{c^2-d^2}}{\sqrt{b^2+d^2}}\right)\right]_{\mathrm{b}}^{\mathrm{a}}.$$

It is plain this expression vanishes except on three occasions; first, when $\phi=\gamma$; second, when $\phi=180^\circ-\gamma$; and, third, when both these conditions are fulfilled, or when $\phi=\gamma=90^\circ$. In the first case its value is

$$2\rho a \text{ arc sin} \frac{a-x'}{\sqrt{(a-x')^2}},$$

which is $\pi\rho a$, if $a-x'$ is positive, but $-\pi\rho a$, if the same quantity is negative. In the second case its value is

$$2\rho a \text{ arc sin} \left[\frac{-a-x'}{\sqrt{2a}}\sqrt{\frac{2a}{(a+x')^2}}\right]=-\pi\rho a.$$

In the third case the sum of these values, or 0, if $a-x'$ is positive, or $-2\pi\rho a$, if it is negative.

The second term of B, after the integration by parts, gives rise to the term

$$2\rho\left[-\frac{2x'^2-a^2-x'x}{3x'^2}\sqrt{b^2+c^2} \text{ arc sin} \frac{\sqrt{c^2-d^2}}{c}\right]_{\mathrm{b}}^{\mathrm{a}}.$$

This term vanishes in all cases except when $a\cos\gamma-\mathrm{a}\cos\phi=0$, or $a\cos\gamma-\mathrm{b}\cos\phi=0$, or the same conditions as obtained in the first term. Therefore, in the first case, the value of this term is

$$-\pi\rho\frac{(2x'+a)(x'-a)}{3x'^2}\sqrt{(x'-a)^2},$$

in the second case,

$$\pi\rho\frac{(2x'-a)(x'+a)^2}{3x'^2},$$

and, in the third case, the sum of these,

$$\frac{2}{3}\pi\rho a\frac{x'^2-a^2}{x'^2} \text{ or } \frac{2}{3}\pi\rho(2x'-a),$$

according as $x'-a$ is positive or negative.

Next attending to Y, if we put

$$P=\frac{\sqrt{c^2-d^2}}{\sqrt{b^2+c^2}}=\frac{\sqrt{(\mathrm{a}-x)(x-\mathrm{b})}}{\sin\phi\sqrt{x'^2+a^2-2x'x}},$$

we have

$$Y=\rho\int_{\mathrm{b}}^{\mathrm{a}}\left[\log\frac{1+P}{1-P}-2P\right]dx,$$

and integration by parts gives the term

$$\left[\rho x\left(\log\frac{1+P}{1-P}-2P\right)\right]_{\mathrm{b}}^{\mathrm{a}}.$$

P always vanishes at both limits, unless at the upper limit when $x' = a$ and $\phi = \gamma$; it cannot vanish at the lower limit as this would necessitate $x' = -a$, which cannot occur with our conventions. Here we are beset with difficulties, but, it seems in this case that we are warranted in regarding $\sqrt{b^2 + c^2}$ as an infinitesimal of a lower order than $\sqrt{c^2 - d^2}$. Thus we shall say that the term introduced by the integration by parts, and which is free from the integrating sign, vanishes in all cases.

Neglecting the discontinuities, and calling B, thus modified, X', we have

$$X' = -2\rho \int_{\mathrm{b}}^{\mathrm{a}} \left\{ x \frac{d}{dx} \left[\text{arc sin} \left(\frac{b}{c} \frac{\sqrt{c^2 - d^2}}{\sqrt{b^2 + d^2}} \right) \right] - \frac{2x'^2 - a^2 - x'x}{3x'^2} \right.$$
$$\left. \times \sqrt{b^2 + c^2} \frac{d}{dx} \left[\text{arc sin} \frac{\sqrt{c^2 - d^2}}{c} \right] \right\} dx,$$

$$Y = -2\rho \int_{\mathrm{b}}^{\mathrm{a}} \left[P + \frac{x}{1 - P^2} \frac{dP}{dx} \right] dx.$$

Then the expressions for X, in its three cases, are

CASE I. $X = 2\pi\rho \left[\pm (x' - \mathrm{b}) - \dfrac{a^3 + x'^3}{3x'^2} \right] + X'.$

CASE II. $X = X'.$

CASE III. $X = 2\pi\rho \left[\mp (\mathrm{a} - x') \pm \dfrac{a^3 - x'^3}{3x'^2} \right] + X'.$

Here the ambiguous sign in Case I must be so read that the quantity may be positive, and in Case III, of the two ambiguous signs, the first must be so read that the quantity may be negative, and the second so read that the term may be positive.

Performing the differentiation with respect to x, indicated in the expression for X', and writing f for $x' \cos\phi - a \cos\gamma$, the first term, after some reductions takes the form

$$2\rho \int_{\mathrm{b}}^{\mathrm{a}} \left[\frac{d \sin\phi\, (bx + c^2)\, x}{c^2} + f \frac{b^2 + c^2}{b^2 + d^2} x \right] \frac{dx}{\sin\phi \sqrt{b^2 + c^2} \sqrt{c^2 - d^2}}.$$

Unless there are discontinuities at the limits, for the factor x, which multiplies the expression under the sign of integration, we may substitute $x - \mathrm{b}$, as the value of the definite integral is not thereby altered.

The second term of X' takes the form

$$2\rho \int_{\mathrm{b}}^{\mathrm{a}} \frac{a\,(2x'^2 - a^2 - x'x)\,(a \cos\phi - x \cos\gamma)\,(b^2 + c^2)}{3x'^2 c^2} \frac{dx}{\sin\phi \sqrt{b^2 + c^2} \sqrt{c^2 - d^2}}.$$

At this point it is desirable to substitute a new variable θ for x, such that

$$x = \mathrm{b} + (\mathrm{a} - \mathrm{b}) \sin^2\theta = a \cos(\phi + \gamma) + 2a \sin\phi \sin\gamma \sin^2\theta.$$

The limits of integration with respect to θ are from 0 to $\frac{\pi}{2}$. If we adopt two constants k and g, such that

$$k^2 = \frac{2(\mathrm{a} - \mathrm{b})x'}{x'^2 + a^2 - 2x'\mathrm{b}} = \frac{4ax' \sin\phi \sin\gamma}{x'^2 + a^2 - 2x'\mathrm{b}},$$

$$g = \frac{\mathrm{a} - \mathrm{b}}{\sin\phi \sqrt{x'^2 + a^2 - 2x'\mathrm{b}}} = \frac{2a \sin\gamma}{\sqrt{x'^2 + a^2 - 2x'\mathrm{b}}},$$

k will be the modulus of the elliptic integrals involved in the problem, and, according to the usual notation, we put $\sqrt{1 - k^2 \sin^2\theta} = \Delta$. Then

$$\frac{dx}{\sin\phi \sqrt{b^2 + c^2} \sqrt{c^2 - d^2}} = \frac{g}{2a \sin\gamma} \frac{d\theta}{\Delta},$$

and

$$X' = \frac{2\rho g}{a \sin\gamma} \int_0^{\frac{\pi}{2}} \left[\frac{(a\cos\gamma - x\cos\phi)(a^2 - x'x)(x - \mathrm{b}) + \frac{a}{3x'^2}(2x'^2 - a^2 - x'x)(a\cos\phi - x\cos\gamma)(b^2 + \mathrm{c}^2)}{c^2} \right.$$
$$\left. + \frac{f(x - \mathrm{b})}{1 - P^2} \right] \frac{d\theta}{\Delta} = \frac{2\rho g}{a \sin\gamma} \int_0^{\frac{\pi}{2}} \left[\frac{Q}{c^2} + f \frac{b^2 + c^2}{b^2 + d^2}(x - \mathrm{b}) \right] \frac{d\theta}{\Delta}.$$

We can suppose

$$\frac{Q}{c^2} = A + A'x + \frac{A''}{a - x} + \frac{A'''}{a + x}.$$

Then

$$(A + A'x)(a^2 - x^2) + A''(a + x) + A'''(a - x) = Q.$$

Making $x = a$, we get

$$2aA'' = 2a^3 \sin^2\frac{\phi + \gamma}{2}(\cos\gamma - \cos\phi)(a - x') + \frac{a^2}{3x'^2}(2x' + a)(x' - a)^3(\cos\phi - \cos\gamma).$$

Making $x = -a$, we get

$$2aA''' = -2a^3 \cos^2\frac{\phi + \gamma}{2}(\cos\gamma + \cos\phi)(x' + a) + \frac{a^2}{3x'^2}(2x' - a)(x' + a)^3(\cos\phi + \cos\gamma).$$

Making $x = 0$, we get

$$a^2 A + a(A'' + A''') = -a^3 \mathrm{b} \cos\gamma + \frac{a^2 \cos\phi}{3x'^2}(2x'^2 - a^2)(x'^2 + a^2).$$

A' is derived by making the division. Thus are obtained the following values:

$$\begin{aligned}
A &= [\tfrac{1}{3}a + x' \cos(\phi + \gamma)]\, a \cos\phi - \frac{a}{3x'}(2x'^2 - a^2)\cos\gamma, \\
A' &= -x' \cos\phi + \tfrac{2}{3} a \cos\gamma, \\
A'' &= a^2(x' - a)\left[\sin^2\frac{\phi+\gamma}{2} + \frac{(2x'+a)(x'-a)^2}{6ax'^2}\right](\cos\phi - \cos\gamma), \\
A''' &= -a^2(x'+a)\left[\cos^2\frac{\phi+\gamma}{2} - \frac{(2x'-a)(x'+a)^2}{6ax'^2}\right](\cos\phi + \cos\gamma).
\end{aligned}$$

In the second portion of X' we have

$$P = \frac{g \sin\theta \cos\theta}{\Delta}.$$

Thus

$$\begin{aligned}
\frac{f(x-\mathrm{b})}{1-P^2} &= 2fa \sin\phi \sin\gamma \frac{(1-k^2\sin^2\theta)\sin^2\theta}{1-(g^2+k^2)\sin^2\theta + g^2\sin^4\theta} \\
&= -2fx' \sin^2\phi + 2fa \sin\phi \sin\gamma \frac{\dfrac{k^2}{g^2} + \left(k'^2 - \dfrac{k^4}{g^2}\right)\sin^2\theta}{1-(g^2+k^2)\sin^2\theta + g^2\sin^4\theta},
\end{aligned}$$

where $k'^2 = 1 - k^2$.

By putting
$$n = -\frac{\sin\phi\sin\gamma}{\sin^2\dfrac{\phi+\gamma}{2}}, \qquad n' = \frac{\sin\phi\sin\gamma}{\cos^2\dfrac{\phi+\gamma}{2}},$$

$$\begin{aligned}
B &= \frac{1}{a}\left[\frac{1}{3}a^2 - 2x'^2\sin^2\phi\right]\cos\phi - \frac{1}{3x'}[2x'^2(1-3\sin^2\phi) \\
&\qquad - 2ax'\cos(\phi+\gamma) - a^2]\cos\gamma, \\
B' &= 2\left[-x'\cos\phi + \frac{2}{3}a\cos\gamma\right]\sin\phi\sin\gamma, \\
B'' &= -(x'-a)\left[\sin^2\frac{\phi+\gamma}{2} + \frac{(2x'+a)(x'-a)^2}{6ax'^2}\right]\frac{\sin\dfrac{\phi-\gamma}{2}}{\sin\dfrac{\phi+\gamma}{2}}, \\
B''' &= -(x'+a)\left[\cos^2\frac{\phi+\gamma}{2} - \frac{(2x'-a)(x'+a)^2}{6ax'^2}\right]\frac{\cos\dfrac{\phi-\gamma}{2}}{\cos\dfrac{\phi+\gamma}{2}},
\end{aligned}$$

we have the expression

$$\begin{aligned}
X' = \frac{2\rho g}{\sin\gamma}\int_0^{\frac{\pi}{2}} \Bigg[& B + B'\sin^2\theta + \frac{B''}{1+n\sin^2\theta} + \frac{B'''}{1+n'\sin^2\theta} \\
& + 2f\sin\phi\sin\gamma \frac{\dfrac{k^2}{g^2} + \left(k'^2 - \dfrac{k^4}{g^2}\right)\sin^2\theta}{1-(g^2+k^2)\sin^2\theta + g^2\sin^4\theta}\Bigg]\frac{d\theta}{\Delta}.
\end{aligned}$$

We next reduce Y to a similar form. The equivalent for this may be written

$$Y = -2\rho \int_0^{\frac{\pi}{2}} \left[P \frac{dx}{d\theta} + \frac{x - \mathrm{b}}{1 - P^2} \frac{dP}{d\theta} \right] d\theta.$$

We have

$$\frac{dP}{d\theta} = g \, \frac{1 - 2 \sin^2 \theta + k^2 \sin^4 \theta}{\Delta^3}.$$

Thus

$$Y = -4\rho a g \sin \phi \sin \gamma \int_0^{\frac{\pi}{2}} \left[2 \sin^2 \theta \cos^2 \theta + \frac{(1 - 2 \sin^2 \theta + k^2 \sin^4 \theta) \sin^2 \theta}{1 - (g^2 + k^2) \sin^2 \theta + g^2 \sin^4 \theta} \right] \frac{d\theta}{\Delta}.$$

SECTION III.—*Removal of the Imaginary Parameters.*

The expressions just derived for X' and Y involve the divisor

$$1 - (g^2 + k^2) \sin^2 \theta + g^2 \sin^4 \theta.$$

The values of k and g are such as make the two parameters resulting from this divisor imaginary. Legendre * has shown how these parameters may be replaced by a single real parameter; but his method is laborious and necessitates the computation of many constants, needless in our case. Fortunately the two integrals concerned are complete, and the modulus k and the two imaginary parameters n and n' satisfy the relation

$$(1 - k)(1 + k) = (1 + n)(1 + n').$$

These circumstances enable us to reduce the elaboration very much.

In the first place let

$$p = \frac{\sin \theta \cos \theta}{\Delta}.$$

Then, by differentiation,

$$\frac{dp}{1 - g^2 p^2} = \frac{1 - 2 \sin^2 \theta + k^2 \sin^4 \theta}{1 - (g^2 + k^2) \sin^2 \theta + g^2 \sin^4 \theta} \frac{d\theta}{\Delta}. \dagger$$

Integrating between the limits 0 and $\frac{\pi}{2}$,

$$\int_0^{\frac{\pi}{2}} \frac{1 - 2 \sin^2 \theta + k^2 \sin^4 \theta}{1 - (g^2 + k^2) \sin^2 \theta + g^2 \sin^4 \theta} \frac{d\theta}{\Delta} = 0.$$

However, it must be noted that there is a discontinuity in the value of this definite integral when $k = 1$. It may be multiplied by a constant at our disposal, and the product joined to either of the previous expressions for X' and Y.

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* Traité des Fonctions Elliptiques, Tom. I, p. 148.

† Traité des Fonctions Elliptiques, Tom. I, p. 72.

In the second place, ζ being a constant, let

$$p = \frac{\sin\theta\cos\theta}{(1+\zeta\sin^2\theta)\,\Delta}.$$

It is plain that, since $\sin\theta\cos\theta$ vanishes at both limits of the integration,

$$\int_0^{\frac{\pi}{2}} \frac{\frac{dp}{d\theta}}{1+\varkappa p^2}\,d\theta = 0,$$

$\varkappa$ being a constant at our disposal. The differentiation being performed,

$$\frac{dp}{1+\varkappa p^2} = \frac{1-(\zeta+2)\sin^2\theta+(2\zeta+1)\,k^2\sin^4\theta-\zeta k^2\sin^6\theta}{(1+\zeta\sin^2\theta)^2\,(1-k^2\sin^2\theta)+\varkappa\sin^2\theta\,(1-\sin^2\theta)}\,\frac{d\theta}{\Delta}.$$

If $\varkappa$ is rightly assumed, the denominator of the first factor of the right member of this can be put under the form

$$[1-(g^2+k^2)\sin^2\theta+g^2\sin^4\theta]\,(1+m\sin^2\theta),$$

provided also that ζ has the value

$$\zeta = -\frac{2g^2}{g^2+k^2}.$$

We have no need to know the value of $\varkappa$, but that of m is

$$m = -\frac{4k^2g^2}{(g^2+k^2)^2} \text{ or } 1+m = \left(\frac{g^2-k^2}{g^2+k^2}\right)^2.$$

It is evident now that we may form the definite integral

$$\int_0^{\frac{\pi}{2}}\left[\frac{1}{\zeta}+\frac{A+A'\sin^2\theta}{1-(g^2+k^2)\sin^2\theta+g^2\sin^4\theta}+\frac{B}{1+m\sin^2\theta}\right]\frac{d\theta}{\Delta} = 0.$$

We may readily satisfy ourselves that the values of A, A' and B are

$$A = 1+\frac{k^2}{g^2}, \qquad A' = -(g^2+k^2)\frac{k^2}{g^2}, \qquad B = \tfrac{1}{2}\left(1-\frac{k^2}{g^2}\right).^*$$

As in the former, there is a discontinuity in the value of this definite integral, when $k=1$. It may be multiplied by a constant at our disposal, and the product joined to either of the preceding expressions for X' and Y.

It is evident that to these constants (four in number) we may give such values that the term under the sign of integration, having the divisor

$$1-(g^2+k^2)\sin^2\theta+g^2\sin^4\theta,$$

will vanish in both cases, and be replaced by terms having the divisor

$$1+m\sin^2\theta.$$

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*I am greatly indebted to Cayley's Elementary Treatise on Elliptic Functions, pp. 127-130, for making the road clear to these values.

Attending first to the modification of X', we may write

$$X' = 2\rho g \sin\phi \int_0^{\frac{\pi}{2}} \left[\Theta + 2f \frac{\frac{k^2}{g^2} + \left(k'^2 - \frac{k^4}{g^2}\right)\sin^2\theta}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta}\right]\frac{d\theta}{\Delta}.$$

Multiply the first definite integral of zero value by f, and the second by $-f$, and add the products to the quantity under the integrating sign. As the result we get

$$X' = 2\rho g \sin\phi \int_0^{\frac{\pi}{2}} \left[\Theta + f\left(\frac{1}{\zeta} - \frac{k^2}{g^2}\right) + \frac{\frac{1}{2}f\left(1 - \frac{k^2}{g^2}\right)}{1 + m\sin^2\theta}\right]\frac{d\theta}{\Delta}.$$

We can simplify the second definite integral by dividing it by $-\frac{1}{\zeta}$, and thus

$$\int_0^{\frac{\pi}{2}} \left[-1 + 2\frac{1 - k^2\sin^2\theta}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta} + \frac{\sqrt{1+m}}{1 + m\sin^2\theta}\right]\frac{d\theta}{\Delta} = 0,$$

where $\sqrt{1+m}$ is to be taken positively. If the first definite integral is multiplied by h, and the second, just given, by H, constants at our disposal, and the products added to the quantity under the integrating sign in the value of Y, we have

$$\int_0^{\frac{\pi}{2}} \left[\frac{(1 - 2\sin^2\theta + k^2\sin^4\theta)(h + \sin^2\theta) + 2H(1 - k^2\sin^2\theta)}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta} + \frac{H\sqrt{1+m}}{1 + m\sin^2\theta} \right.$$
$$\left. - H + 2\sin^2\theta\cos^2\theta\right]\frac{d\theta}{\Delta}$$

Here h and H must be so chosen that the first fraction of this shall become integral. Then it is plain the expression will assume the form

$$\int_0^{\frac{\pi}{2}} \left[h + H + \left(2 + \frac{k^2}{g^2}\right)\sin^2\theta - 2\sin^4\theta + \frac{H\sqrt{1+m}}{1 + m\sin^2\theta}\right]\frac{d\theta}{\Delta}.$$

The equations, which determine h and H, are

$$g^2(g^2 + k^2 - 2)h + 2g^4 H = k^2 - g^2,$$
$$g^2(g^2 - k^2)h + 2g^4 H = k^2(g^2 + k^2) - 2g^2,$$

whence are derived

$$h = -\frac{g^2 + k^2}{2g^2}, \quad H = \frac{1}{4g^4}\left[(g^2 + k^2)^2 - 4g^2\right].$$

Thus

$$Y = -4\rho a g \sin\phi \sin\gamma \int_0^{\frac{\pi}{2}} \left[\frac{1}{4g^4}(g^4 - k^4 + 4g^2) + \left(2 + \frac{k^2}{g^2}\right)\sin^2\theta - 2\sin^4\theta \right.$$
$$\left. + \frac{H\sqrt{1+m}}{1 + m\sin^2\theta}\right]\frac{d\theta}{\Delta}.$$

SECTION IV.—*Evaluation of the Elliptic Integrals by Means of the Θ Function.*

The readiest method of arriving at the numerical values of X' and Y is by the use of the Θ function of Jacobi. For the parts which involve the first and second complete functions of Legendre, we have

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\Delta} = K, \quad \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\Delta} d\theta = \frac{1}{k^2}(K - E'),$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{\Delta} d\theta = \frac{1}{3k^2}(2K - E') - \frac{2}{3k^4}(K - E').$$

K and E' can be derived from the equations

$$k = \sin\psi, \quad \frac{q + q^9 + q^{25} + \dots}{1 + 2(q^4 + q^{16} + q^{36} + \dots)} = \left(\frac{\sin\frac{1}{2}\psi}{1 + \sqrt{\cos\psi}}\right)^2, \quad \sqrt{\cos\psi} = \chi,$$

$$q = \frac{1}{2}\,\frac{1-\chi}{1+\chi} + 2\left(\frac{1}{2}\,\frac{1-\chi}{1+\chi}\right)^5 + 30\left(\frac{1}{2}\,\frac{1-\chi}{1+\chi}\right)^9 + \dots,$$

$$K = \frac{\pi}{2}\left(\frac{2}{1+\sqrt{\cos\psi}}\right)^2 [1 + 2(q^4 + q^{16} + q^{36} + \dots)]^2,$$

$$K - E' = \frac{\pi}{2}\,\frac{(1+\sqrt{\cos\psi})^8}{\sqrt{\cos\psi}}\,\frac{q - 4q^4 + 9q^9 - 16q^{16} + \dots}{[1 + 2(q^4 + q^{16} + q^{36} + \dots)]^8}.$$

These formulæ have the inconvenience of approximating the indeterminate form $\frac{0}{0}$ as k becomes smaller. This may be removed by employing the relation

$$k^2 = 16q\left[\frac{1 + q^2 + q^6 + q^{12} + \dots}{1 + 2(q + q^4 + q^9 + \dots)}\right]^4.$$

Consequently

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\Delta} d\theta = \frac{\pi}{32}\,\frac{(1+\sqrt{\cos\psi})^8}{\sqrt{\cos\psi}}\,\frac{1 - 4q^3 + 9q^8 - 16q^{15} + \dots}{[1 + 2(q^4 + q^{16} + q^{36} + \dots)]^8} \times \left[\frac{1 + 2(q + q^4 + q^9 + \dots}{1 + q^2 + q^6 + q^{12} + \dots)}\right]^4.$$

In case k is quite small, we may employ the series in powers of k,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{\Delta} d\theta = \frac{\pi}{6}\left[\left(\frac{1}{2}\right)^2\frac{3}{2} + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\frac{4}{3}k^2 + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\frac{5}{4}k^4 + \dots\right].$$

The similar series for the second definite integral is

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\Delta} d\theta = \frac{\pi}{2}\left[\frac{1}{2} + \left(\frac{1}{2}\right)^2\frac{3}{4}k^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\frac{5}{6}k^4 + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\frac{7}{8}k^6 + \dots\right].$$

The complete elliptic integrals of the third kind of Legendre have the general form

$$\int_0^{\frac{\pi}{2}} \frac{A}{1+n\sin^2\theta}\,\frac{d\theta}{\Delta},$$

where n may be positive or negative. Consider first the case where it is negative. Here, if k^2 exceeds $-n$, we derive a from the equation (using the Gudermannian notation)

$$\operatorname{sn}(a)=\frac{\sqrt{-n}}{k},$$

and we have

$$\int_0^{\frac{\pi}{2}} \frac{A}{1+n\sin^2\theta}\,\frac{d\theta}{\Delta}=AK\left[1+\sqrt{\frac{-n}{(k^2+n)(1+n)}}\,\frac{\Theta'(a)}{\Theta(a)}\right],$$

where

$$Z(a)=\frac{\Theta'(a)}{\Theta(a)}=\frac{\pi}{2K}\,\frac{4q\sin\left(\frac{\pi}{K}a\right)-8q^4\sin 2\left(\frac{\pi}{K}a\right)+12q^9\sin 3\left(\frac{\pi}{K}a\right)-\ldots}{1-2q\cos\left(\frac{\pi}{K}a\right)+2q^4\cos 2\left(\frac{\pi}{K}a\right)-2q^9\cos 3\left(\frac{\pi}{K}a\right)+\ldots}.\text{*}$$

If either one or both of $-n$, k are unity, this definite integral is infinite. But, as the attraction of the segment cannot be infinite, it follows that the multiplier A must, in this case, vanish, and the product be finite.

If n is positive we derive a from the equation (i is put for $\sqrt{-1}$)

$$\operatorname{sn}(ia)=\frac{\sqrt{-n}}{k},$$

and we have

$$\int_0^{\frac{\pi}{2}} \frac{A}{1+n\sin^2\theta}\,\frac{d\theta}{\Delta}=AK\left[1+i\sqrt{\frac{n}{(k^2+n)(1+n)}}\,\frac{\Theta'(ia)}{\Theta(ia)}\right].$$

But

$$i\frac{\Theta'(ia)}{\Theta(ia)}=-\frac{\pi}{K}\left[\frac{q^{1-b}}{1-q^{1-b}}-\frac{q^{1+b}}{1-q^{1+b}}+\frac{q^{3-b}}{1-q^{3-b}}-\frac{q^{3+b}}{1-q^{3+b}}+\ldots\right],$$

where $b=\frac{a}{K'}$, K' being the complementary period.†

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* Jacobi, Fundamenta Nova, p. 184.

† Journal für die Mathematik, Vol. XXXIX, p. 345.

The third case where $-n$ exceeds k^2 can only have place for the parameters n and m. If either of the equations

$$1 + n = \frac{\sin^2 \frac{\phi - \gamma}{2}}{\sin^2 \frac{\phi + \gamma}{2}} = k'^2 \sin^2 \lambda, \quad 1 + m = \left(\frac{g^2 - k^2}{g^2 + k^2}\right)^2 = k'^2 \sin^2 \lambda,$$

affords a real value for λ, the corresponding elliptic integral must be eliminated and replaced by another having a different parameter, by means of the relation

$$\frac{1+n}{n} \int_0^{\frac{\pi}{2}} \frac{1}{1 + n \sin^2 \theta} \frac{d\theta}{\Delta} - \frac{1 - n''}{n''} \int_0^{\frac{\pi}{2}} \frac{1}{1 - n'' \sin^2 \theta} \frac{d\theta}{\Delta} = \frac{k^2}{nn''} K,$$

where n'' is determined by the equation

$$(1 + n)(1 - n'') = k'^2.*$$

SECTION V.—*Investigation of the Cases of Indetermination in the Expressions for X' and Y.*

From the value we have given for k^2, it is plain that if, for the moment, M denote the greatest distance of the attracted point from the edge of the segment and N the least, we will have the equation

$$k^2 = \frac{M^2 - N^2}{M^2}.$$

Thus $k = 0$ when the point is on the axis of the segment, and $k = 1$ when it is on the edge; and a value intermediate in other situations

The definite integrals

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\Delta} \text{ and } \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta}{\Delta} d\theta$$

are infinite for $k = 1$. The third integral

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta \cos^2 \theta}{\Delta} d\theta$$

is not infinite. The other three definite integrals of Legendre's third kind, involved in the question, are to the parameters,

$$n = -\frac{\sin \phi \sin \gamma}{\sin^2 \frac{\phi + \gamma}{2}}, \quad n' = \frac{\sin \phi \sin \gamma}{\cos^2 \frac{\phi + \gamma}{2}}, \quad m = -\frac{4ax' \sin \phi \sin \gamma}{(x' \sin \phi + a \sin \gamma)^2}.$$

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* Traité des Fonctions Elliptiques, Tom. I, p. 72.

The first integral is infinite when $\phi = \gamma$; the second when $\phi = \gamma$ and $x' = a$; the third when $\phi = \gamma$ and $x' = a$, or when $x' \sin \phi = a \sin \gamma$. When $x' = a$ the attracted point is on the sphere; if $f = x' \cos \phi - a \cos \gamma = 0$, it is on the plane of the base; if both conditions are fulfilled it is on the edge of the segment. The third condition $x' \sin \phi = a \sin \gamma$ has been introduced only by the elimination of the imaginary parameters. To make the discontinuities and indeterminations clear, it seems necessary to consider apart each case.

CASE I.—When $\phi = \gamma$.

Here we have

$$k^2 = \frac{4ax' \sin^2 \gamma}{x'^2 - 2ax' \cos 2\gamma + a^2}, \quad g^2 = \frac{4a^2 \sin^2 \gamma}{x'^2 - 2ax' \cos 2\gamma + a^2},$$

and, putting

$$B = \frac{1}{3x'}(a^2 + 3ax' - 2x'^2) + \left(-\frac{2x'^2}{a} + 2x' - \frac{4}{3}a\right)\sin^2 \gamma,$$

$$B' = 2\left(-x + \frac{2}{3}a\right)\sin^2 \gamma,$$

$$B''' = -(x' + a)\left[1 - \frac{(2x' - a)(x' + a)^2}{6ax'^2}\sec.^2 \gamma\right],$$

we have

$$X' = \mp \pi\rho a\left[1 + \frac{(2x' + a)(x' - a)^2}{3ax'^2}\right] + 2\rho g \cot \gamma \int_0^{\frac{\pi}{2}}\left[B + B' \sin^2 \theta\right.$$

$$\left. + \frac{B'''}{1 + \tan^2 \gamma \sin^2 \theta} + 2(x' - a)\sin^2 \gamma \frac{\frac{k^2}{g^2} - \left[k'^2 - \frac{k^4}{g^2}\right]\sin^2 \theta}{1 - (g^2 + k^2)\sin^2 \theta + g^2 \sin^4 \theta}\right]\frac{d\theta}{\Delta}.$$

The upper of the ambiguous signs must be read when $x' - a$ is positive, the lower when it is negative.

CASE II.—When $x' = a$.

Here we have

$$k^2 = \frac{\sin \phi \sin \gamma}{\sin^2 \frac{\phi + \gamma}{2}} = -n, \quad g^2 = \frac{\sin^2 \gamma}{\sin^2 \frac{\phi + \gamma}{2}},$$

and, putting

$$B = \left[\frac{1}{3} - 2\sin^2\phi\right]\cos\phi - \left[\frac{1}{3} - 2\sin^2\phi - \frac{2}{3}\cos(\phi+\gamma)\right]\cos\gamma,$$

$$B' = 2\left[-\cos\phi + \frac{2}{3}\cos\gamma\right]\sin\phi\,\sin\gamma,$$

$$B''' = 2\left[\frac{2}{3} - \cos^2\frac{\phi+\gamma}{2}\right]\frac{\cos\frac{\phi-\gamma}{2}}{\cos\frac{\phi+\gamma}{2}},$$

we have

$$X' = \mp\pi\rho a + \frac{2\rho g a}{\sin\gamma}\int_0^{\frac{\pi}{2}}\Bigg[B + B'\sin^2\theta + \frac{B'''}{1 + n'\sin^2\theta}$$
$$+ 2(\cos\phi - \cos\gamma)\sin\phi\,\sin\gamma\,\frac{\frac{k^2}{g^2} + \left[k'^2 - \frac{k^4}{g^2}\right]\sin^2\theta}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta}\Bigg]\frac{d\theta}{\Delta}.$$

CASE III.—When $\phi = \gamma$ and $x' = a$.

Here we have

$$k^2 = 1, \quad g^2 = 1, \quad f = 0, \quad n' = \tan^2\gamma,$$

and, putting

$$B = \frac{2}{3} - \frac{4}{3}\sin^2\gamma, \quad B' = -\frac{2}{3}\sin^2\gamma, \quad B''' = -2\left(1 - \frac{2}{3}\sec^2\gamma\right),$$

we have

$$X' = \mp\pi\rho a + 2\rho a\cot\gamma\int_0^{\frac{\pi}{2}}\left[B + B'\sin^2\theta + \frac{B'''}{1 + \tan^2\gamma\sin^2\theta}\right]\frac{d\theta}{\cos\theta}.$$

In deriving this expression, it has been assumed that we might write $x - \mathrm{b}$ for x in a certain place, without thereby altering the value of the integral. But this is not permissible when the discontinuities have undergone a change. If we had written $x - a$ instead and neglected the discontinuity $\mp\pi\rho a$, the result would have been correct. To avoid the hap-hazard character of this course of proceeding, we return to the original expression for X, before integration by parts was employed.

Throwing out the factor $(a - x)^2$, common in this case to the numerators and denominators of the two terms, we have

$$X' = 2\rho\int_{\mathrm{b}}^{a}\left[-\operatorname{arc\,sin}\sqrt{\frac{x - \mathrm{b}}{a + x}} + \sqrt{\frac{a - x}{2a}}\operatorname{arc\,sin}\left(\frac{1}{\sin\gamma}\sqrt{\frac{x - \mathrm{b}}{a + x}}\right)\right]dx.$$

Employing integration by parts so as to bear on the arcs, we have, as the part freed from the sign of integration, the expression

$$-2\rho a \operatorname{arc\,sin}\sqrt{\frac{a-\mathrm{b}}{2a}}+2\rho\mathrm{b}\operatorname{arc\,sin}\sqrt{\frac{\mathrm{b}-\mathrm{b}}{a+\mathrm{b}}}.$$

Now the second term of this vanishes except possibly when $a+\mathrm{b}=0$, or in the case when the segment is a hemisphere. But if the hemisphere has an attraction continuous with the attraction of the segments adjacent to it, it is plain we must assume that $\frac{\mathrm{b}-\mathrm{b}}{a+\mathrm{b}}=0$ even when $a+\mathrm{b}=0$. The value of the first term is $-2\rho a\gamma$, and, joining to this the portion still under the sign of integration, we have

$$X'=-2\rho a\gamma+\frac{\sqrt{2a}}{3}\rho\cos\gamma\int_{\mathrm{b}}^{a}\left[1+\frac{a}{a+x}\right]\frac{dx}{\sqrt{x-\mathrm{b}}}.$$

Put $\sqrt{x-\mathrm{b}}=u$, then

$$\begin{aligned}X'&=-2\rho a\gamma+\tfrac{2}{3}\sqrt{2a}\rho\cos\gamma\int_{0}^{\sqrt{2a}\sin\gamma}\left[1+\frac{a}{a+\mathrm{b}+u^2}\right]du\\&=-2\rho a\gamma+\tfrac{2}{3}\sqrt{2a}\rho\cos\gamma\left[\sqrt{2a}\sin\gamma+\frac{a}{\sqrt{2a}\cos\gamma}\gamma\right]\\&=\tfrac{4}{3}\rho a(\sin\gamma\cos\gamma-\gamma)\end{aligned}$$

In case $\gamma=\frac{\pi}{2}$, we have $X'=-\frac{2}{3}\pi\rho a$, which agrees with the known expression for the attraction of the homogeneous sphere for a point on its surface.

To determine the value of Y for the present case, we observe, that taking P for the independent variable, we have, for this case,

$$dx=4a\sin^2\gamma\,.\,PdP\,;$$

thus

$$\begin{aligned}Y&=4\rho a\sin^2\gamma\int_0^1\left[\log\frac{1+P}{1-P}-2P\right]PdP\\&=8\rho a\sin^2\gamma\int_0^1\left[\tfrac{1}{3}P^4+\tfrac{1}{5}P^6+\tfrac{1}{7}P^8+\ldots\right]dP.\\&=8\rho a\sin^2\gamma\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots\right].\end{aligned}$$

The last factor of this is a convergent series, although it is the difference of two divergent series; and it is easy to see that its value is $\frac{1}{6}$. Then

$$Y = \frac{4}{3}\rho a \sin^2\gamma.$$

Calling Y the lateral attraction, the formula shows that this is proportional to the square of the sine of the angular radius of the segment; and, in the case of the hemisphere, this quantity is obtained by dividing the downward attraction of the whole sphere by π.

CASE IV.—When $x' \sin\phi = a\sin\gamma$.

Here we have the multipliers of the infinite definite integral

$$1 - \frac{k^2}{g^2} = 0, \quad \sqrt{1+m} = 0,$$

Therefore we must determine the limits towards which converge the two expressions

$$\int_0^{\frac{\pi}{2}} \frac{1 - \frac{k^2}{g^2}}{1 - \frac{4k^2 g^2}{(g^2+k^2)^2}\sin^2\theta}\,\frac{d\theta}{\Delta} \text{ and } \int_0^{\frac{\pi}{2}} \frac{\sqrt{1+m}}{1 + m\sin^2\theta}\,\frac{d\theta}{\Delta},$$

when m approaches -1. But we have, in this case,

$$1 - \frac{k^2}{g^2} = \pm\, 2\sqrt{1+m},$$

positive if k^2 is smaller than g^2, and negative if it is greater. Hence we need consider only the latter expression. The radical $\sqrt{1+m}$ being taken positively, at the limit $m = -1$, the value of the integral is $\frac{\pi}{2}\frac{1}{k'}$.* If k is unity this is infinite; but this case has already been treated.

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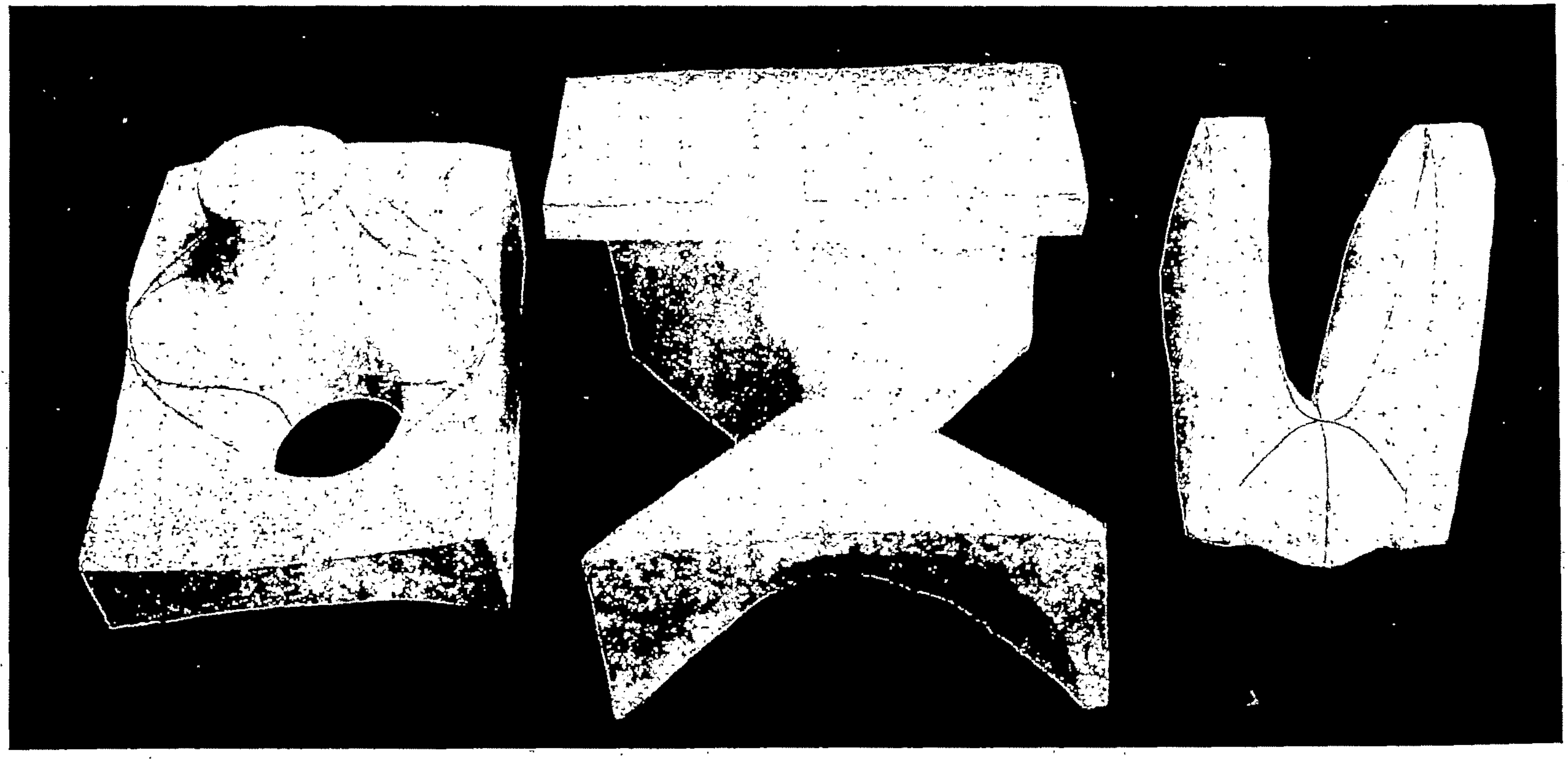
* This may easily be deduced from the equation (m') of Legendre, Traité des Fonctions Elliptiques, Tom. I, p. 138.



THE CARDIOID SURFACE

THE DELTOID SURFACE

TRANSLATION SURFACE
CONNECTED WITH THE CUSPIDAL CUBIC

On a Certain Class of Algebraic Translation-Surfaces.

BY JOHN EIESLAND.

In the following pages have been discussed types of translation-surfaces that are generated in four different ways. This class of surfaces owes its creation to S. Lie, who developed the general theory and also began a discussion of certain special cases. The important theorem by which he connected the parametric representation of such surfaces with the well-known theorem of Abel Lie always considered as one of his greatest achievements, and no one who reads his memoir in the Leipziger Berichte of 1892 can fail to admire the wonderful penetration of his genius. Two of Lie's students, R. Kummer and Georg Wiegner, have carried on the study of special cases, and Georg Scheffer, also one of his students, has in the Acta Mathematica, Vol. 24, given an admirable resumé of Lie's work and also an independent and elegant treatment of certain parts of the calculations. Recently Paul Stäckel, in the American Transactions, Vol. 7, has presented a paper on the minimal surfaces belonging to this class, in which the treatment is independent of Abel's theorem. It is to be hoped that his method will be fruitful in his further investigation along the same line. G. Scheffer, in the above mentioned paper, expresses his opinion that an extended and detailed treatment of this class of surfaces would be interesting not only from the standpoint of surface-theory, but also ought to be undertaken because of the bearing this subject has on the theory of functions.

As the field is very large and the number of types are very many, I have in this paper limited myself to algebraic surfaces only, and the following investigations will show that this limitation is not arbitrary, but is rather the most natural one in the discussion of the subject, inasmuch as I have shown that *the quartic curve which determines such surfaces must be unicursal and have no double points with distinct tangents.*

I have thought it advisable to introduce the subject by giving a short resumé of facts known long ago concerning translation-surfaces and statements of a few well known theorems.

I.

If a curve C_0, having a point p_0 in common with another curve C_1, is translated parallel to itself, any point on it will describe a curve C which is congruent with C_1 and similarly placed. The surface generated by C_0 may also be considered as generated by C and is called a *translation-surface.* It follows then that *a translation-surface may be generated in at least two different ways.*

The general parametric representation of such a surface is

$$x = A(u) + A_1(v), \quad y = B(u) + B_1(v), \quad z = C(u) + C_1(v), \tag{1}$$

whose generating curves are

$$x = A(u), \quad y = B(u), \quad z = C(u),$$

and

$$x = A_1(v), \quad y = B_1(v), \quad z = C_1(v).$$

A translation-surface may also be generated as follows: Consider the two curves

$$x = 2A(u), \quad y = 2B(u), \quad z = 2C(u),$$
$$x = 2A_1(v), \quad y = 2B_1(v), \quad z = 2C_1(v).$$

If we join any point on the first curve to a point on the second by a straight line, the locus of the middle points of this chord will evidently be the translation-surface

$$x = A(u) + A_1(v), \quad y = B(u) + B_1(v), \quad z = C(u) + C_1(v),$$

on which the generating curves are congruent to the given curves and similarly placed, but drawn on one-half the scale.* This definition is due to Lie.

Since the linear tangent at a point p on a (u) curve is moved parallel to itself, when this curve is translated along a (v) curve, the ensemble of all these parallel tangents forms a cylinder which is tangent to the surface, and hence, *the* (u) *and* (v) *lines are conjugate lines.*†

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*See Darboux, Leçons, Vol. I, p. 99.

†S. Lie, Beiträge zur Theorie der Minimalflächen (Math. Ann. t. XIV, pp. 332-387). See also Darboux, Leçons, Vol. I, p. 103.

If we construct the tangents to a curve C_1, we obtain a developable surface which cuts out a curve on the plane at infinity. Since all the curves C_1 are congruent and similarly placed, their respective developables will all cut out one and the same curve; the same holds for the curves C_0. Now suppose that to a tangent at a point on C_1 corresponds a parallel tangent at a point on C_0, or, what amounts to the same thing, the tangents to both curves are parallel to the elements of the same irreducible cone. In this case the two developables belonging to C_1 and C_0 respectively will cut out one and the same curve on the plane at infinity; moreover, *these curves have a common envelope*; for, since the tangents to the curves C_0 are parallel to each other at all points where they are cut by C_1, and since by hypothesis the tangents along C_1 and C_0 are parallel, there must exist on each curve C_1 a point p where the curve is touched by the intersecting curve C_0; the locus of all points p is a common envelope Σ of the curves C_1 and C_0; in fact, when the curve C_1 by translation passes over into the next consecutive curve, the fixed point p moves along the curve C_0, that is along C_1 itself. The surface may therefore be conceived as generated by a translation of the curve C_1 in such a way that it always touches the envelop Σ, or by translation of C_0 in the same manner.* The curve Σ is an asymptotic line, since the tangents along it are conjugate to themselves.

Suppose, for example, that the translation-surface is represented by the equations

$$x = A(u) + A(v), \qquad y = B(u) + B(v), \qquad z = C(u) + C(v), \qquad (2)$$

which may be obtained from (1) by letting A_1, B_1, C_1 be the same functions of (v) as A, B, C are of (u). In this case the above condition is satisfied, C_0 and C_1 being congruent. It is evident that the envelope Σ is obtained by putting $u = v$ in (2), so that the equation of this curve becomes

$$x = 2A(u), \qquad y = 2B(u), \qquad z = 2C(u),$$

which is an asymptotic line similar to C_0 and C_1 and drawn on twice the scale. *The surface (2) may therefore be considered as the locus of the middle points of all chords of Σ; the curves C_0 and C_1 form a single irreducible family of congruent and similarly placed curves.*

In 1872 Lie proposed and solved the problem to find all translation-surfaces which may be generated by translation in an infinite number of ways.[*] These surfaces belong to a more general class that are generated by translation in four different ways. In 1882 Lie completely determined all such surfaces.[†] Later he noticed that this problem had some connection with Abel's theorem applied to curves of the fourth order, but it was not till 1892 that, after repeated efforts, he was able to state and prove the following important theorem :

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* For proof of this theorem see Lie-Scheffer, Berührungstransformationen, p. 362.

If on a translation-surface that can be generated in more than two ways we draw tangents at any point along the four generating curves, the intersection of these tangents with the plane at infinity is a curve of the fourth order.

Conversely, if we suppose given in the plane at infinity a curve of the fourth order, there exist always infinitely many (∞^4) *surfaces generated in four ways, whose tangents along the generating curves intersect the plane at infinity in points situated on the given curve.*

The co-ordinates of these surfaces are expressible as a sum of any two Abelian Integrals with respect to the four points of intersection of any variable straight line with the quartic.

Every direction in space is determined by a point in the plane at infinity. Let x, y, z be a point in space and $x + dx$, $y + dy$, $z + dz$ the consecutive point; the direction of a line joining these two points is completely determined whenever the ratios $\frac{dx}{dz}$, $\frac{dy}{dz}$ are given; we may therefore with Lie consider these ratios as coordinates ξ, η in the plane at infinity.[§]

Suppose now given in this plane a quartic curve $F(\xi, \eta) = 0$; in order to determine the translation-surface, according to Lie's theorem, we form the Abelian integrals of the first kind

$$\Phi = \int \frac{\xi \, d\xi}{F'_{(\eta)}}, \qquad \Psi = \int \frac{\eta \, d\eta}{F'_{(\eta)}}, \qquad \mathrm{X} = \int \frac{d\xi}{F_{(\eta)}}.$$

The limits of these integrals we fix in the following manner: We suppose the quartic cut by a fixed and a variable straight line; denoting the abscissas of the points of intersection by $\xi_1^0, \xi_2^0, \xi_3^0, \xi_4^0$, and $\xi_1, \xi_2, \xi_3, \xi_4$ respectively, we choose the former as the lower and the latter as the upper limits so that we have,

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[*] Kurzes Requmé mehrerer neuen Theorien, Ges. d. w. zu Kristiania, May 8d, 1872. Archiv for Math. og Naturvidenskab, B. 4, 1879.

[†] Archiv for Math. og Nat. B. 7, Kristiania 1882.

[‡] Compte Rendue, B. 114 (1893, p. 334-337).

[§] Lie-Scheffer, Berührtr. p. 357.

$$\Phi_i = \int_{\xi_i^0}^{\xi_i} \frac{\xi_i \, d\xi_i}{F'_{(\eta_i)}}, \quad \Psi_i = \int_{\xi_i^0}^{\xi_i} \frac{\eta_i \, d\eta_i}{F'_{(\eta_i)}}, \quad X_i = \int_{\xi_i^0}^{\xi_i} \frac{d\xi_i}{F'_{(\eta_i)}}. \qquad (i = 1, 2, 3, 4.)$$

Now by Abel's theorem we have

$$\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = 0,$$
$$\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 = 0,$$
$$X_1 + X_2 + X_3 + X_4 = 0,$$

from which it follows that

$$\Phi_1 + \Phi_2 = -\Phi_3 - \Phi_4,$$
$$\Psi_1 + \Psi_2 = -\Psi_3 - \Psi_4,$$
$$X_1 + X_2 = -X_3 - X_4,$$

so that the equations

$$x = \Phi_1 + \Phi_2, \qquad y = \Psi_1 + \Psi_2, \qquad z = X_1 + X_2 \qquad (3)$$

represent the same surface as

$$x = -\Phi_3 - \Phi_4, \qquad y = -\Psi_3 - \Psi_4, \qquad z = -X_3 - X_4, \qquad (4)$$

a translation-surface generated in four ways, as is seen from the double mode of representation.

If the quartic is irreducible, the integrals Φ_1, Φ_2, Φ_3, and Φ_4 have the same form, and likewise the Ψ's and X's; the surface (3), or (4), has therefore the same property as the surface (2), *i. e.* $\xi_1 = \xi_2$ is an envelope of the generating curves $\xi_1 =$ const., $\xi_2 =$ const., and $\xi_3 = \xi_4$ an envelope of the curves $\xi_3 =$ const., $\xi_4 =$ const. Moreover the surface is symmetrical with respect to the origin as is evident from equations (3) and (4); *it has therefore a centre.*

A linear projective transformation leaves the plane at infinity at rest; any curve in this plane will therefore be transformed projectively into a curve in the same plane, while the corresponding translation-surface will be transformed into a translation-surface. A given linear projective transformation of the surface determines a projective transformation of the curve $F(\xi, \eta) = 0$. If, on the other hand, a projective transformation of $F(\xi, \eta) = 0$ is given, the linear transformation of the surface is not uniquely determined; for, let the transformation be,

$$\xi_1 = \frac{a_1 \xi + a_2 \eta + a_3}{c_1 \xi + c_2 \eta + c_3}, \qquad \eta_1 = \frac{b_1 \xi + b_2 \eta + b_3}{c_1 \xi + c_2 \eta + c_3},$$

where

$$\xi_1 = \frac{dx_1}{dz_1}, \quad \eta = \frac{dy_1}{dz_1}, \quad \xi = \frac{dx}{dz}, \quad \eta = \frac{dy}{dz},$$

and hence,

$$dx_1 = \rho(a_1\,dx + a_2\,dy + a_3\,dz),$$
$$dy_1 = \rho(b_1\,dx + b_2\,dy + b_3\,dz),$$
$$dz_1 = \rho(c_1\,dx + c_2\,dy + c_3\,dz),$$

from which we obtain the transformation

$$x_1 = \rho(a_1 x + a_2 y + a_3 z) + a_4,$$
$$y_1 = \rho(b_1 x + b_2 y + b_3 z) + b_4,$$
$$z_1 = \rho(c_1 x + c_2 y + c_3 z) + c_4,$$

where ρ_1, a_4, b_4, c_4 are entirely arbitrary, q. e. d. Thus, suppose given the identical transformation $\xi_1 = \xi$, $\eta_1 = \eta$, which leaves the quartic unaltered; we have,

$$\begin{aligned} a_1 &= 1, & a_2 &= 0, & a_3 &= 0, \\ b_1 &= 0, & b_2 &= 1, & b_3 &= 0, \\ c_1 &= 0, & c_2 &= 0, & c_3 &= 1, \end{aligned}$$

and the corresponding transformation in space becomes

$$x_1 = \rho x + a_4, \qquad y_1 = \rho y + b_4, \qquad z_1 = \rho z + c_4;$$

hence, *to the same quartic curve in the plane at infinity belong* ∞^4 *translation-surfaces.*

All the translation-surfaces that are projectively equivalent with respect to a linear projective transformation are said to belong to the same *type*. If therefore we wish to obtain a certain type we consider all the projectively equivalent quartics in the plane at infinity and form the Abelian integrals with respect to all such curves.

The quartic curve in the plane at infinity may degenerate as follows:

(1) In a cubic and a straight line,
(2) In two conics,
(3) In a conic and two straight lines,
(4) In four straight lines.

The three last cases have been studied by R. Kummer in a thesis published in 1894.*

When the quartic degenerates into two conics two essentially different cases may present themselves.† Let the conics be C_1 and C_2 and let the variable line cut them in four points P_1, P_2, P_3, P_4. If now p be a point on the surface, two things may happen:

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* Die Flächen mit unendlichvielen Erzeugenden durch Translation von Curven, Inaugural-Dissertation von Richard Kummer; Leipzig, 1894.

† See Lie-Scheffer, Berührtr. B. 1, p. 409.

1°. The point P_1 and P_2, viz., the points of intersection of the conjugate tangents $p\,P_1$, $p\,P_2$ with the plane at infinity, lie on the same conic, or,

2°. They lie on different conics.

In the first case it may be proved by means of Desargue's theorem that the corresponding surfaces contain ∞^1 congruent translation-curves corresponding to the family of ∞^1 conics determined by the two given conics;[*] *the corresponding surface is therefore generated in an infinite number of ways.*

In the second case we obtain surfaces generated in four ways only.[†] The general equation of these surfaces is

$$Ae^{z+y} + Be^{z+x} + Ce^{x+y} + Le^{x} + Me^{y} + Ne^{z} = 0. \tag{5}$$

To this class belong also all the transforms of (5) by linear transformations and their degenerates obtained by special choice of the conics C_1 and C_2.

In the first and more important case we obtain the surfaces

$$Ae^{x} + Be^{y} + Ce^{z} + D = 0 \tag{6}$$

and their transforms by linear projective transformations.[‡]

If one of the conics is the imaginary circle $\xi^2 + \eta^2 + 1 = 0$, Scherk's minimal surface

$$e^{z} = \frac{\sin x}{\sin y}$$

and its transforms are obtained; this corresponds to the case where the family of conics is determined by four imaginary points. If two of these points are real we obtain the transcendental surface

$$e^{z} = \frac{e^{2x} + 1}{-\,2e^{x}\sin y}.$$

If the four points are the I and J points counted twice, that is to say, the family of conics are concentric circles, we get the helicoid

$$z = \arctan\frac{y}{x}.$$

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[*] For proof see Lie-Scheffer, Berührtr., p. 406.

[†] p. 364 ibid.

[‡] By linear projective transformations we mean here as elsewhere the following:

$$x' = a_1x + b_1y + c_1z + d_1, \qquad y' = a_2x + b_2y + c_2z + d_2, \qquad z' = a_3x + b_3y + c_3z + d_3.$$

Finally, if all the four points coincide, forming a point of contact of the third order, Cayley's ruled surface is obtained.*

The case where the quartic degenerates into a cubic and a straight line has been treated in a thesis by Georg Wiegner (1893).† He arrives at the important result that *when the straight line is an inflexional tangent to the cubic the corresponding translation-surface has a family of parabolae as generators.* He tried to prove the converse of this theorem, but failed for the very good reason that it is not true. In fact, we shall prove that *in the case of a cuspidal cubic and its cuspidal tangent the same thing happens: A surface belongs to it which has a set of parabolae as generators and which is algebraic and of the fourth degree, while the only algebraic surface that Wiegner obtained is of the third degree.*‡

By a projective transformation a cuspidal cubic may be brought into the form

$$\eta^2 - \xi^3 = 0,$$

whose cuspidal tangent is $\eta = 0$. Putting now

$$F(\xi, \eta) = (\eta^2 - \xi^3)\,\eta = 0$$

and forming the Abelian integrals of the first kind with respect to this quartic, we have, since $F'_{\eta_1} = 2\xi_1^3$ and $F'_{\eta_2} = -2\xi_2^3$,

$$X = \tfrac{1}{2}\int \frac{\xi_1\,d\,\xi_1}{\xi_1^3} - \int \frac{\xi_2\,d\,\xi_2}{\xi_2^3} = -\tfrac{1}{2}\,\xi_1^{-1} + \xi_2^{-1} + \text{const.}$$

$$Y = \tfrac{1}{2}\int \frac{\xi_1^{3/2}\,d\,\xi_1}{\xi_1^3} - \int \frac{\eta_2\,d\,\xi_2}{\xi_2^3} = -\xi_1^{-\frac{1}{2}} + \text{const.},$$

$$Z = \tfrac{1}{2}\int \frac{d\,\xi_1}{\xi_1^3} - \int \frac{d\,\xi_2}{\xi_2^3} = -\tfrac{1}{4}\,\xi_1^{-2} + \tfrac{1}{2}\,\xi_2^{-2} + \text{const.}$$

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* Such is the case with all the conics of the family $\xi^2 + k\eta^2 + 3\eta = 0$, which touch at (0, 0) and have the same curvature at this point.

† Über eine besondere Klasse von Translationsflächen, Inaugural-Dissertation von Georg Wiegner, Separatausdruck aus dem Archiv for Math. og Naturv., B. 16, 1893.

‡ This surface is $zx^2 - 2yz = zx$. Georg Scheffer in his admirable paper "Das Abel'sche Theorem und das Lie'sche Theorem über Translationsflächen" (Acta Math, Vol. 28) has repeated Wiegner's mistake when he says: "Und nur in diesem Fall (viz., a cubic and its inflexional tangent) treten Parabeln als erzeugende Curven auf." See p. 90, ibid.

Putting const. = 0 and transforming, these equations may be written

$$X = \frac{1}{\xi_1} - \frac{2}{\xi_2},$$

$$Y = \frac{1}{\sqrt{\xi_1}},$$

$$Z = \frac{1}{\xi_1^2} - \frac{2}{\xi_2^2},$$

from which we obtain the quartic surface

$$2Z = Y^4 + 2Y^2X - X^2, \tag{7}$$

the generating parabolæ of which are all parallel to the parabola $X^2 + 2Z = 0$. The surface may therefore be generated by letting the vertex of this parabola be translated along the quartic space curve

$$X = \frac{1}{\xi_1}, \quad Y = \frac{1}{\sqrt{\xi_1}}, \quad Z = \frac{1}{\xi_1^2},$$

the plane of the parabolæ being always kept parallel to the YZ-plane. The intersection of the surface with any plane $Z = c$ is a quartic curve which, when $Z = 0$, degenerates into two parabolæ, $Y^4 + 2Y^2X - X^2 = 0$, tangents to each other at the origin, which thus becomes a saddle-point. (See plate.) The second pair of generating curves may now be obtained. We put

$$X = -\frac{1}{\xi_3} - \Phi_1,$$

$$Y = -\frac{1}{\sqrt{\xi_3}} - \Phi_2,$$

$$Z = -\frac{1}{\xi_3^2} - \Phi_3,$$

and substitute in the equation (7), which is satisfied if $\Phi_1 = \Phi_2^2$ and $\Phi_3 = \Phi_2^4$; hence we have as second mode of representation

$$X = -\frac{1}{\xi_3} - \frac{1}{\xi_4},$$

$$Y = -\frac{1}{\sqrt{\xi_3}} - \frac{1}{\sqrt{\xi_4}},$$

$$Z = -\frac{1}{\xi_3^2} - \frac{1}{\xi_4^2}.$$

If we put $\xi_3 = \xi_4$ we obtain the envelope $X^2 + 2Z = 0$ which, as was explained above, is an asymptotic curve on the surface.* It should also be noticed that the family of asymptotic curves on (7) may be obtained by quadrature according to Lie, since it admits of the following projective transformation into itself

$$\bar{X} = \lambda^2 X, \quad \bar{Y} = \lambda Y, \quad \bar{Z} = \lambda^4 Z.$$

We shall now restate Wiegner's theorem as follows:

When a quartic breaks down into either a cubic and its inflexional tangent, or a cubic and its cuspidal tangent, the corresponding translation-surfaces have a family of parabolae as generating curves.†

To this may also be added the following theorem, which from our standpoint is important:

There exist two types of algebraic surfaces corresponding to

1°. *A cuspidal cubic and its cuspidal tangent.*

2°. *A cuspidal cubic and its inflexional tangent.*

When the quartic breaks down into two conics C_1 and C_2, and if we choose the points P_1 and P_2 on C_1 and C_2 respectively, we obtain, as was stated above, surfaces of the general form

$$Ae^{y+z} + Be^{z+x} + Ce^{x+y} + Le^x + Me^y + Ne^z = 0.$$

By special choice of the conics these surfaces may degenerate *and even become algebraic.* Two real conics being given, the following six cases may occur:

The two conics may intersect in

1°. Four real and distinct points.

2°. Two real and two imaginary points.

3°. Four imaginary points.

4°. Two pairs of consecutive points.

5°. Four real points of which three are consecutive.

6°. Four consecutive points.

In all cases except the last we obtain transcendental surfaces.

We shall not prove this proposition, as it involves rather tedious calculations, taking up one case after another and forming the corresponding Abelian integrals in each case. We shall therefore take up the last case and study the corresponding surfaces.

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* This curve is drawn on the model. See plate.

† The converse of this theorem has been omitted; it may easily be proved.

Let the conics C_1 and C_2 have four consecutive points in common. By a projective transformation we may always transform them in such a way that the point of contact coincides with the vertices of the conics. A family of such conics is represented by the equation

$$\eta^2 + k\xi^2 - \xi = 0.$$

Taking any two values of k, say k_1 and k_2, we write the quartic

$$F(\xi\eta) = (\eta^2 + k_1\xi^2 - \xi)(\eta^2 + k_2\xi^2 - \xi) = 0,$$

or, homogeneously,

$$F(\xi, \eta, \zeta) = (\eta^2 + k_1\xi^2 - \xi\zeta)(\eta^2 + k_2\xi^2 - \xi\zeta) = 0,$$

which by means of the transformation

$$\eta = \eta', \qquad \xi = \xi', \qquad \zeta = \zeta' + \tfrac{1}{2}(k_1 + k_2)\xi'$$

may be brought into the form

$$[\eta^2 + \tfrac{1}{2}(k_1 - k_2)\xi^2 - \xi\zeta]\,[\eta^2 - \tfrac{1}{2}(k_1 - k_2)\xi^2 - \xi\zeta] = 0.$$

If further we put $\sqrt{\tfrac{1}{2}(k_1 - k_2)}\,\xi = \xi'$, and $\zeta = \sqrt{\tfrac{1}{2}(k_1 - k_2)}\,\zeta'$, we obtain the following simple form for the two conics, putting $\zeta' = 1$:

$$F(\xi, \eta) = (\eta^2 + \xi^2 - \xi)(\eta^2 - \xi^2 - \xi) = 0.$$

We have now

$$F'_{\eta_1} = 4\xi_1^2\eta_1, \qquad F'_{\eta_2} = -4\xi_2^2\eta_2,$$

and forming the Abelian integrals we obtain the following parametric representation of the surface:

$$X = \tfrac{1}{4}\int\frac{d\xi_1}{\xi_1\sqrt{\xi_1^2 + \xi_1}} - \tfrac{1}{4}\int\frac{d\xi_2}{\xi_2\sqrt{\xi_2 - \xi_2^2}},$$

$$Y = \tfrac{1}{4}\int\frac{d\xi_1}{\xi_1^2} - \tfrac{1}{4}\int\frac{d\xi_2}{\xi_2^2},$$

$$Z = \tfrac{1}{4}\int\frac{d\xi_1}{\xi_1^2\sqrt{\xi_1^2 + \xi_1}} - \tfrac{1}{4}\int\frac{d\xi_2}{\xi_2^2\sqrt{\xi_2 - \xi_2^2}}.$$

Integrating and using the transformations

$$-4X = X', \qquad -4Y = Y', \qquad -4Z = Z', \qquad \frac{1}{\xi_1} = u^2 - 1, \qquad \frac{1}{\xi_2} = v^2 + 1,$$

we obtain

$$X = u - v, \qquad Y = u^2 - v^2 - 2, \qquad Z = u^3 - v^3 - 3u - 3v. \tag{8}$$

Eliminating u and v we obtain the surface

$$4ZX = X^4 + 3Y^2 - 12 = 0,$$

which may be easily thrown into the form

$$ZX = X^4 + Y^2 - 1, \tag{8'}$$

a quartic surface *which is symmetrical with respect to the origin.* The planes $X =$ const. are parabolic sections which for $X = 0$ degenerate into two straight lines $Y = \pm 1$. The locus of the vertices of these parabolæ is a plane quartic obtained by putting $Y = 0$ in (8′). *The generating curves are twisted cubics,* as is easily seen for equations (8).

To the same reducible quartics belongs also the surface obtained from (8′) by means of the imaginary transformations

$$X = iX', \qquad Y = iY', \qquad Z = iZ,$$

that is, the surface

$$ZX = -X^4 - Y^2 - 1,$$

which differs from (8′) by having two separate sheets, the plane $X = 0$ being asymptotic to the surface. We shall state the above result thus: *To a reducible quartic consisting of two conics having four consecutive points in common there corresponds a type of algebraic translation-surfaces, generated in four distinct ways, which are reducible to the form*

$$ZX = X^4 + Y^2 - 1.$$

II.

We shall now take up the study of algebraic surface connected with an irreducible quartic, and we propose to solve the problem of finding all the types of algebraic surfaces corresponding to such a curve. The results obtained will be interesting not only from the standpoint of surface-theory but also from that of theory of functions. The Abelian integral connected with a unicursal curve has naturally no proper period; it becomes what Poincaré calls a degenerate Abelian integral possessing poles and logarithmic singularities.* The following investigation may therefore be considered as a study of such integrals.

If the quartic is of genus 1, its co-ordinates are expressible as functions of a parameter θ and $\sqrt{\Theta}$ where Θ is of the fourth degree in θ. The co-ordinates

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* Sur les surface de translation et les fonctions Abeliennes, Bulletin de la Societé Math., v. 29 (1901), pp. 61-86.

X, Y, Z of the corresponding surface are therefore elliptic integrals and the surface is transcendental. It is evident that quartics of genus 2 and 3 also give rise to the same kind of surfaces. We may therefore limit ourselves to unicursal curves, in which case the co-ordinates are expressible as rational functions of a parameter. The co-ordinates of the surface thus become integrals of rational functions. The problem is now to determine the cases in which these integrals themselves become algebraic functions. We shall prove the following important

Theorem. Whenever the unicursal quartic in the plane at infinity has at least one double point with distinct tangents the corresponding surface is transcendental.

Let the quartic have one double point and two cusps. By means of a proper projective transformation it may be thrown into the form

$$y^2 + x^2 - 2xy + x^2 y^2 - 2x^2 y + 2kxy^2 = 0, \tag{9}$$

having a cusp at the origin and at the end of the x-axis, while the double point is at the end of the y-axis. A parametric representation of this curve is obtained by determining each point on it as the intersection of the curve with a variable conic passing through the three singular points and another fixed point on the curve.* We shall choose as our conic the hyperbola

$$xy + \rho x + \sigma y = 0,$$

and as the fixed point the intersection of the cuspidal tangent at the origin with the curve. It is evident that the following relation between ρ and σ must exist:

$$\rho + \sigma = 2(k - 1).$$

Substituting now $x = \dfrac{-\sigma y}{\rho + y}$ in (9) and reducing, we easily find the co-ordinates of the variable point, viz:

$$x = \frac{2(1-k)}{(\rho+1)^2},$$

$$y = \frac{2(1-k)}{\rho^2 + 2(2-k)\rho + 5 - 4k},$$

which will serve as a parametric representation of the quartic. Substituting these values in

$$F'_y = 2(y - x + x^2 y - x^2 + 2kxy)$$

we obtain, after some reduction, the following simple expression:

$$F'_y = \frac{-8(1-k)}{(\rho+1)^3}.$$

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* See Salmon, Higher Plane Curves, p. 250.

We now form the Abelian integrals and obtain the surface

$$\begin{aligned}X&=\int\frac{d\rho_1}{(\rho_1+1)^2}+\int\frac{d\rho_2}{(\rho_2+1)^2},\\ Y&=\int\frac{d\rho_1}{\rho_1^2+2(2-k)\rho_1+5-4k}+\int\frac{d\rho_2}{\rho_2^2+2(2-k)\rho_2+5-4k},\\ Z&=\frac{1}{2(1-k)}\int d\rho_1+\frac{1}{2(1-k)}\int d\rho_2,\end{aligned}$$

or, performing the integration,

$$\begin{aligned}X&=-\frac{1}{\rho_1+1}-\frac{1}{\rho_2+1},\\ Y&=\frac{1}{\sqrt{1-k^2}}\tan^{-1}\frac{\rho_1+2-k}{\sqrt{1-k^2}}+\frac{1}{\sqrt{1-k^2}}\tan^{-1}\frac{\rho_2+2-k}{\sqrt{1-k^2}},\\ Z&=\frac{1}{2(1-k)}\rho_1+\frac{1}{2(1-k)}\rho_2,\end{aligned}$$

which is evidently a transcendental surface, provided k is an arbitrary constant differing from $+1$ or -1 Q. E. D.

When $k=1$ the quartic breaks down into two coincident hyperbolæ, hence this value of k must be excluded.

If $k=-1$ the curve becomes a tricuspidal quartic or *deltoid*;[*] we may therefore obtain the corresponding surface by putting $k=1$ in (9) and integrating. We thus obtain the algebraic surface

$$\begin{aligned}X&=-\frac{1}{\rho_1+1}-\frac{1}{\rho_2+1},\\ Y&=-\frac{1}{\rho_1+3}-\frac{1}{\rho_2+3},\\ Z&=\tfrac{1}{4}\rho_1+\tfrac{1}{4}\rho_2.\end{aligned}\tag{10}$$

Putting $X=-X'$, $Y=-Y'$, $4Z=Z'$ we obtain by eliminating ρ_1 and ρ_2

$$Z(2XY+Y-X)=6X-2Y-8XY;$$

and transforming this surface to the centre of symmetry which is $X=\frac{1}{2}$, $Y=-\frac{1}{2}$, $Z=4$, we reduce the equation to the simpler form

$$Z(2XY+\tfrac{1}{2})=2(X+Y)\tag{11}$$

which is a cubic surface *whose sections parallel to the three co-ordinate planes are*

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[*] I have called the corresponding surface *the deltoid surface*. I have made a model of this surface, which, together with others, are found in the collection of the Johns Hopkins University, Baltimore.

hyperbolae. In the finite portion of space the following right lines are seen to lie on the surface:

(1) $Z=0,\ X+Y=0$; (2) $Z=2,\ X=\frac{1}{2}$; (3) $Z=2,\ Y=\frac{1}{2}$;
(4) $X=0,\ Z=4Y$; (5) $Y=0,\ Z=4X$; (6) $X=\frac{1}{2},\ Y=-\frac{1}{2}$;
(7) $X=-\frac{1}{2},\ Y=\frac{1}{2}$, (8) $Z=-2,\ X=-\frac{1}{2}$; (9) $Z=-2,\ Y=-\frac{1}{2}$.

The locus of the vertices of the parabolae $Z=$ const. is a plane cubic. In order to find its equation it will be convenient to transform the X and Y-axis by turning them through an angle of 45°; we obtain then

$$Z(X^2-Y^2+\tfrac{1}{2})=\sqrt{2}X.$$

Putting $Y=0$, we have

$$Z(X^2+\tfrac{1}{2})=\sqrt{2}\,X,$$

a cubic curve having the X — axis as asymptote, and a maximum and minimum at $Z=\pm 1$, $X=\pm\frac{1}{2}\sqrt{2}$, corresponding to the two saddle-points on the surface.

The translation-curves are twisted cubics, as is easily seen from the parametric representation of the surface (11) which is

$$X=\frac{1}{\rho_1+1}+\frac{1}{\rho_2+1}+\tfrac{1}{2},$$
$$Y=\frac{1}{\rho_1+3}+\frac{1}{\rho_2+3}-\tfrac{1}{2},$$
$$Z=\rho_1+\rho_2+4.$$

The envelope of these translation-curves is

$$X=\frac{2}{\rho_1+1}+\tfrac{1}{2},$$
$$Y=\frac{2}{\rho_1+3}-\tfrac{1}{2},$$
$$Z=2\rho_1+4.$$

We shall state the results of the above development as follows:

The necessary condition that a translation-surface that can be generated in four ways is algebraic, is that the corresponding quartic in the plane at infinity is unicursal and possesses no double point with distinct tangents.[*]

Before we discuss the cases where the unicursal quartic has a tac-node, osc-node, or triple point, we shall study a rather interesting case of the surface (11) belonging to the same type.

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[*] The following investigations will show that this is also the sufficient condition.

If we apply to the quartic (9) the linear imaginary transformation

$$x = x' + iy', \qquad y = x' - iy'$$

we obtain the cardioid

$$(x^2 + y^2 - 2x)^2 = 4\,(x^2 + y^2).$$

This transformation places the two cusps at infinity at the imaginary *I* and *J* points. The parametric representation of this curve is

$$x = \frac{4\,(1 - \rho^2)}{(\rho^2 + 1)^2}, \quad y = \frac{8\rho}{(\rho^2 + 1)^2}.$$

We have now

$$dx = \frac{8\rho\,(\rho^2 - 3)}{(\rho^2 + 1)^3}, \quad F'_y = \frac{64\rho\,(3 - \rho^2)}{(1 + \rho^2)^3},$$

and the equation of the corresponding surface is

$$X = -\tfrac{1}{2}\int \frac{(1 - \rho_1^2)}{(1 + \rho_1^2)}\,d\rho_1 - \tfrac{1}{2}\int \frac{(1 - \rho_2^2)}{(1 + \rho_2^2)^2}\,d\rho_2 = -\tfrac{1}{2}\,\frac{\rho_1}{1 + \rho_1^2} - \tfrac{1}{2}\,\frac{\rho_2}{1 + \rho_2^2},$$

$$Y = -\int \frac{\rho_1\,d\rho_1}{(1 + \rho_1^2)^2} - \int \frac{\rho_2\,d\rho_2}{(1 + \rho_2^2)^2} = -\frac{1}{1 + \rho_1^2} - \frac{1}{1 + \rho_2^2},$$

$$Z = -\tfrac{1}{8}\int d\rho_1 - \tfrac{1}{8}\int d\rho_2 = -\tfrac{1}{8}\,\rho_1 - \tfrac{1}{8}\,\rho_1.$$

Eliminating ρ_1 and ρ_2 we obtain the equation

$$4Z\,(4X^2 + Y^2 - Y) = X,$$

which by a suitable real and linear transformation may be thrown into the form

$$Z(X^2 + Y^2 - 1) = 2X.^* \tag{12}$$

A plane $Z =$ const. cuts out a circle whose equation is

$$\left(X - \frac{1}{c}\right)^2 + Y^2 = 1 + \frac{1}{c^2}.$$

For $Z = \infty$ we obtain the unit circle and for $Z = 0$ a straight line $X = 0$. The locus of centres of these circles is the hyperbolae $XZ = 1$; the plane $Y = 0$ intersects the surface in the cubic curve $Z(X^2 - 1) = 2X$ which has a point of inflexion at the origin and the Z-axis as asymptote. Any plane $Z = mX$ cuts the surface in a family of ∞^1 circles whose planes all pass through the Y-axis.

To the same cardioid also belongs the surface obtained from (12) by using the transformation

$$X = iX', \qquad Y = iY', \qquad Z = iZ';$$

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* This surface I have called *The Cardioid-Surface.* See Plate.

the resulting cubic surface

$$Z(X^2 + Y^2 + 1) + 2X = 0 \qquad (12')$$

differs from (12) in several respects, although it belongs to the same type. The plane $Z = \infty$ intersects it in the imaginary circle $X^2 + Y^2 + 1 = 0$, and the planes $Z = \pm 1$ is tangent to the surface at the two umbilical points $(-1, 0, 1)$, $(1, 0, -1)$.

In conclusion it should be noticed that all the surfaces belonging to this type have a real double point at infinity.

III.

Quartics with Node-Cusp and Ac-Node Cusp.

We shall now consider the case where the quartic has an ordinary cusp and a tac-node. We shall prove the following theorem:

In order that the surface shall be algebraic, the tac-node must degenerate into a node-cusp.

To a quartic with a node-cusp and an ordinary cusp corresponds an algebraic translation-surface of the fourth degree.

A quartic with a tac-node and a cusp may by a projective transformation be thrown into the form

$$x^3 y + ax^4 + (1 - a)\,yx^2 - y^2 = 0,$$

in which the tac-node is placed at the origin and the cusp at the end of the y-axis. For the sake of convenience we shall put $a = b - 1$, so that the parametric representation may be written

$$x = \frac{\rho^2 + b\rho}{\rho + 1}, \quad y = \frac{(\rho^2 + b\rho)^2}{\rho + 1}.$$

We have also,

$$F'_y = x^3 + (2 - b)\,x^2 - 2y = -\frac{(\rho^2 + b\rho)^2}{(\rho + 1)^3}\,[\rho^2 + (2 - b)\,\rho + b]$$

and

$$dx = \frac{\rho^2 + (2 - b)\,\rho + b}{(\rho + 1)^2}.$$

Forming now the Abelian integrals we have

$$X = -\int \frac{d\rho_1}{\rho_1^2 + b\rho_1} - \int \frac{d\rho_2}{\rho_2^2 - b\rho_2},$$

$$Y = -\int d\rho_1 - \int d\rho_2,$$

$$Z = -\int \frac{(\rho_1 + b)\,d\rho_1}{(\rho_1^2 + b\rho_1)^2} - \int \frac{(\rho_2 + b)\,d\rho_2}{(\rho_2^2 + b\rho_2)^2},$$

or, integrating,

$$\begin{aligned} X &= -\frac{1}{b}\log\frac{\rho_1}{\rho_1+1} - \frac{1}{b}\log\frac{\rho_2}{\rho_2+1}, \\ Y &= -\rho_1 - \rho_2, \\ Z &= -\frac{1}{b^2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) + \frac{b-2}{b^3}\log\frac{\rho_1\rho_2}{(\rho_1+1)(\rho_2+1)}, \end{aligned}$$

which is a transcendental surface for $b =$ *any arbitrary constant differing from* 0. Q. E. D.

If $b = 0$ the quartic reduces to the form

$$x^3y - x^4 + 2yx^2 - y^2 = 0,$$

which has a node-cusp at the origin. The corresponding surface is

$$\begin{aligned} X &= -\frac{1}{\rho_1} - \frac{1}{\rho_2}, \\ Y &= -\rho_1 - \rho_2, \\ Z &= -\tfrac{1}{2}\left(\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}\right) - \tfrac{1}{3}\left(\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right), \end{aligned} \tag{13}$$

or, eliminating,

$$Y(3X^2 + 2X^3 - 6Z) + 6X^2 + 6X = 0.$$

Transforming this surface to its centre of symmetry $(-\tfrac{1}{2}, 0, \tfrac{1}{12})$, we have

$$Y(2X^3 - \tfrac{3}{2}X - 6Z) + 6(X^2 - \tfrac{1}{4}) = 0,$$

which may be still further simplified by putting $\tfrac{2}{3}Y = Y'$, $X = X'$, $3X + 12Z = 4Z'$, so that we finally obtain

$$Y(4X^3 - 4Z) + 4X^2 - 1 = 0, \tag{13'}$$

which is a quartic surface whose plane sections parallel to the YZ-plane are equilateral hyperbolae. The translation-curves are twisted quartics as are also their two envelopes $\rho_1 = \rho_2$, $\rho_3 = \rho_4$. For $X = \pm\tfrac{1}{2}$ the sections become two pairs of intersecting lines. We shall not enter into a detailed study of this surface, but proceed to the next case.

When three consecutive nodes coincide, we have what is called an osc-node. If we consider such a node as formed by the coincidence of a tac-node with a double point, it will seem evident from the principle of continuity and from the preceding development that also in this case will the surface be transcendental. We shall, however, prove this more rigorously thus:

A quartic with an osc-node at the origin may be written (See Salmon, Higher Plane Curves, p. 249, sec. ed., § 290)

$$(y - mx^2)^2 + cxy\,(y - x^2) + dy^3 + gx^2y^2 + hxy^3 + iy^4 = 0,$$

or, introducing homogeneous co-ordinates,

$$(yz - mx^2)^2 + cxy\,(yz - x^2) + dy^3z + gx^2y^2 + hxy^3 + iy^4 = 0.$$

We now choose the vertices of the triangle of reference as follows: Let $x = 0$, $y = 0$ be the osc-node, $y = 0$ being the tangent at this point. As second vertex, $y = 0$, $z = 0$, we take a point of inflexion on the curve, making $z = 0$ the inflexional tangent. The third vertex will then be uniquely determined at some point which in general does not lie on the curve. The equation will now have the form

$$(yz - mx^2)^2 + cxy\,(yz - mx^2) + dy^3z = 0,$$

in which the coefficients d and m may be reduced to unity by putting $x = \frac{x'}{\sqrt{dm}}$, $y = \frac{y'}{d}$, $z = z'$, so that we finally obtain the simple form

$$(y - x^2)^2 + cxy\,(y - x^2) + y^3 = 0.\text{*} \tag{14}$$

A parametric representation of this curve may now be obtained as follows:

Writing (14) in the form

$$(y - x^2)^2 + cxy\,(y - x^2) + y^2\,(y - x^2) + x^2y^2 = 0,$$

we see that the curve is closely connected with the conic

$$z'^2 + cx'z' + y'z' + x'^2 = 0,\dagger$$

which may be represented parametrically by putting

$$x' = -\sigma\left(t + \frac{c}{2}\right), \quad y' = \sigma\left(\frac{c^2}{4} - 1 - t^2\right), \quad z' = \sigma;$$

but since $x' : y' : z' = xy : y^2 : y - x^2$, we have

$$xy = -\sigma\left(t + \frac{c}{2}\right), \quad y^2 = \sigma\left(\frac{c^2}{4} - 1 - t^2\right), \quad y - x^2 = \sigma.$$

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[*] If d x m have opposite signs we put $x = \frac{x'}{\sqrt{-gm}}$, $y = \frac{y}{d}$, $z = z'$ which reduces the quartic to the form

$$(y + x^2)^2 + cxy\,(y + x^2) + y^3 + 0.$$

[†] See Salmon, Higher Plane Curves, p. 249 sec. ed.

Solving these equations for x and y we find

$$x = \frac{\left(t + \frac{c}{2}\right)\left(t^2 + \frac{c^2}{4} - 1\right)}{ct + \frac{c^2}{2} - 1}, \quad y = \frac{\left(t^2 - 1 + \frac{c^2}{4}\right)^2}{ct + \frac{c^2}{2} - 1}.$$

Calculating now F'_y in terms of t and forming the Abelian integrals we have the surface

$$\begin{aligned}
X &= -\int \frac{\left(t_1 + \frac{c}{2}\right) dt_1}{\left(\frac{c^2}{4} - 1\right)^2 - t_1^4} - \int \frac{\left(t_2 + \frac{c}{2}\right) dt_2}{\left(\frac{c^2}{4} - 1\right)^2 - t_2^4}, \\
Y &= -\int \frac{dt_1}{\frac{c^2}{4} - 1 - t_1^2} - \int \frac{dt_2}{\frac{c^2}{4} - 1 - t_2^2}, \\
Z &= \int \frac{\left(ct_1 + \frac{c^2}{2} - 1\right) dt_1}{\left(\frac{c^2}{4} - 1 + t_1^2\right)^2 \left(\frac{c^2}{4} - 1 - t_1^2\right)} - \int \frac{\left(ct_2 + \frac{c^2}{2} - 1\right) dt_2}{\left(\frac{c^2}{4} - 1 + t_2^2\right)^2 \left(\frac{c^2}{4} - 1 - t_2^2\right)},
\end{aligned} \tag{15}$$

which is transcendental provided c differs from 2 or -2. Q. E. D.

If $c = \pm 2$ the surface (15) is algebraic and only then. In these cases the quartic has a tac-node-cusp at the origin. Now the case $c = -2$ may be reduced to that of $c = +2$ by a projective transformation. We have therefore the theorem:

To a quartic with a tac-node-cusp corresponds the algebraic surface

$$\begin{aligned}
X &= -\int \frac{t_1 + 1}{t_1^3} dt_1 - \int \frac{t_2 + 1}{t_2^4} dt_2 = \tfrac{1}{2}\left(\frac{1}{t_1^2} + \frac{1}{t_2^2}\right) + \tfrac{1}{3}\left(\frac{1}{t_1^3} + \frac{1}{t_2^3}\right), \\
Y &= -\int \frac{dt_1}{t_1^2} - \int \frac{dt_2}{t_2^2} = \frac{1}{t_1} + \frac{1}{t_2}, \\
Z &= -\int \frac{2t_1 + 1}{t_1^6} dt_1 - \int \frac{2t_2 + 1}{t_2^6} dt_2 = \tfrac{1}{2}\left(\frac{1}{t_1^4} + \frac{1}{t_2^4}\right) + \tfrac{1}{5}\left(\frac{1}{t_1^5} + \frac{1}{t_2^5}\right),
\end{aligned}$$

and its transforms by linear projective transformations. If now we put $\frac{1}{t_1} = u$, $\frac{1}{t_2} = v$ and interchange Y and Z we have

$$\begin{aligned}
X &= \tfrac{1}{2}(u^2 + v^2) + \tfrac{1}{3}(u^3 + v^3), \\
Y &= \tfrac{1}{2}(u^4 + v^4) + \tfrac{1}{5}(u^5 + v^5), \\
Z &= u + v,
\end{aligned}$$

or, eliminating,

$$(1+Z)Y=(1+Z)\left(\frac{Z^5}{5}+\frac{Z^4}{2}\right)-(Z^3+2Z^2)\left(\frac{Z^2}{2}+\frac{Z^3}{3}-X\right)+\left(\frac{Z^2}{2}+\frac{Z^3}{3}-X\right)^2.$$

Transforming this surface to its centre of symmetry ($\frac{1}{3}, -\frac{1}{5}$, 1) we have

$$ZY+\frac{1}{45}Z^6-\frac{1}{12}Z^4-\frac{1}{3}XZ^3-X^2+XZ+\frac{1}{2}Z^2+\frac{13}{36}=0,$$

which may be simplified by putting

$$\tfrac{1}{2}Z+Y+X=Y', \qquad X=X' \qquad Z=Z',$$

so that we have

$$ZY+\frac{1}{45}Z^6-\frac{1}{12}Z^4-\frac{1}{3}XZ^3-X^2+\frac{13}{36}=0, \tag{16}$$

a sextic translation-surface whose plane sections parallel to the XY-plane are parabolae, which for $Z=0$ become two parallel straight lines $X=\pm\sqrt{\frac{13}{36}}$; the surface resembles therefore (8′). The translation-curves are twisted quintics. The surface is unique in this respect, that its degree is the highest of any algebraic translation-surface of the kind we are considering.

IV.

Quartics with a Triple-Point.

A quartic with a triple-point may be written

$$z(ay^3+by^2x+cyx^2+dx^3)=kx^4+lx^3y+mx^2y^2+nxy^3+py^4.$$

We shall suppose that the tangents at the triple-point are all real and distinct. For the sake of convenience we shall place the triple-point at $z=0$, $x=0$ and then write the equation in non-homogeneous form

$$y(a+bx+cx^2+dx^3)=kx^4+lx^3+mx^2+nx+p.$$

Now, since $F'_y=a+bx+cx^2+dx^3$, we have

$$\begin{aligned}
X&=\int\frac{x_1\,dx_1}{a+bx_1+cx_1^2+dx_1^3}+\int\frac{x_2\,dx_2}{a+bx_2+cx_2^2+dx_2^3},\\
Y&=\int\frac{(kx_1^4+lx_1^3+mx_1^2+nx_1+p)\,dx_1}{(a+bx_1+cx_1^2+dx_1^3)^2}+\int\frac{(kx_2^4+lx_2^3+mx_2^2+nx_2+p)\,dx_2}{(a+bx_2+cx_2^2+dx_2^3)^2},\\
Z&=\int\frac{dx_1}{a+bx_1+cx_1^2+dx_1^3}+\int\frac{dx_2}{a+bx_2+cx_2^2+dx_2^3},
\end{aligned} \tag{17}$$

which is a transcendental surface. If, however, the tangents at the triple-point coincide, this surface becomes algebraic and only in this case, as is easily seen, by putting $a + bx + cx^2 + dx^3 = d(x - \alpha)^3$ in (17). We may therefore say

To a quartic having a triple-point with coincident tangents correspond algebraic translation-surfaces.

We shall now consider these surfaces and find the simplest forms to which they can be reduced.

We write the quartic

$$yx^3 = kx^4 + lx^3 z + mx^2 z^2 + nxz^3 + pz^4;$$

interchanging the x and z-coordinates this may be written non-homogeneously

$$y = k + lx + mx^2 + nx^3 + px^4.$$

By a projective transformation this equation may be reduced to one of the following two forms:

$$\begin{aligned} &(a)\quad y = x^2 + x^4, \\ &(b)\quad y = x^4. \end{aligned} \tag{18}$$

Since the form (a) is not projectively equivalent to (b), *there will be two types of surfaces*, the second type being connected with a quartic which in addition to the triple-point also has a point of undulation. We shall consider the case (a) first. The corresponding surface may be written

$$\begin{aligned} (2)\quad X &= \int x_1\,dx_1 + \int x_2\,dx_2 = x_1^2 + x_2^2, \\ Y &= \int (x_1^2 + x_1^4)\,dx_1 + \int (x_2^2 + x_2^4)\,dx_2 = \tfrac{1}{3}(x_1^2 + x_2^2) + \tfrac{1}{5}(x_1^5 + x_2^5), \\ Z &= \int dx_1 + \int dx_2 = x_1 + x_2. \end{aligned}$$

By elimination and transformation we get

$$Y = \frac{Z}{5}\left(\frac{5X^4 - Z^4}{4}\right) + \frac{Z}{6}(3X - Z^2),$$

whose centre of symmetry is easily found to be (1, 0, 0). Transforming to this point as origin we have,

$$Y = \frac{ZX^2}{4} - \frac{Z^3}{6} - \frac{Z^5}{20},$$

which by a homothetic transformation may be thrown into the form

$$Y = ZX^2 - Z^5 - Z^3, \tag{19}$$

a quintic surface resembling the surface (12) very closely, the difference being that instead of having circular section parallel to the XY-plane it has parabolic sections. To the same curve belong also the real surfaces obtained from (19) by means of the imaginary transformation $X = iX'$, $Y = iY'$, $Z = iZ'$; we obtain the surface

$$Y = Z^3 - Z^5 - ZX^2. \tag{19'}$$

The intersection of the plane $Y = 0$ with the surface (19) is the line $Z = 0$ and the quartic $X^2 = Z^4 + Z^2$, which has a double point at the origin, the two branches going to infinity in all four quadrants. In (19′) the same plane cuts out $X^2 = Z^2 - Z^4$, which is a lemniscate. The curves cut out by the plane $X = 0$ are, in both cases quintic curves with a point of inflexion at the origin; but in the case of the surface (19) this quintic intersects the Z-axis in two real points outside of the origin and symmetrically situated with respect to it, while in the case of (19′) these two points are imaginary.

Case (*b*). *The quartic* $y = x^4$,

We have the surface

$$(2)\quad X = \int x_1\,dx_1 + \int x_2\,dx_2 = x_1^2 + x_2^2,$$

$$Y = \int x_1^4\,dx_1 + \int x_2^4\,dx_2 = \tfrac{1}{5}(x_1^5 + x_2^5),$$

$$Z = \int dx_1 + \int dx_2 = x_1 + x_2,$$

or, transforming and eliminating,

$$Y = \frac{Z}{5}(5X^2 - Z^4), \tag{20}$$

which by a homothetic transformation may be reduced to the form

$$Y = ZX^2 - Z^5.$$

The locus of vertices of the parabolic sections is the curve $Y = -Z^5$, and the plane $Y = 0$ cuts out the straight line $Z = 0$ and two parabolae $X = \pm Z^2$. The surface resembles (19); its transform by the imaginary transformation used above intersects the plane $Y = 0$ in two imaginary parabolae. It should also be noticed that the asymptotic lines of (20) may be found by quadratures, according to a general theorem of Lie.

We shall now give a resumé of the preceding results as follows:

PLANE AT INFINITY, $(x\ y)$	SPACE OF THREE DIMENSIONS, (X, Y, Z)
Quartic Curve.	*Translation-surfaces that can be Generated in Four Distinct Ways.*
I. Two Conics; $(y^2+x^2-x)(y^2-x^2-x)=0$	I. $ZX=X^4+Y^2-1$
II. A unicursal cubic and its inflexional tangent: $(y-x^3)y=0$	II. $ZX^2-2YZ=2X$ (Wiegner's surface)
III. A unicursal cubic and its cuspidal tangent: $(y^2-x^3)y=0$	III. $2Z=Y^4+2Y^2X-X^2$
IV. Non-reducible unicursal quartics: (*a*) tricuspidal quartic, (*deltoid*) (*b*) *cardioid*	IV. *The deltoid surface*: (*a*) $Z(2YZ+\frac{1}{2})=2(X+Y)$ (*b*) *The cardioid surface*: $Z(X^2+Y^2-1)=2X$
V. Quartic with a node cusp and an ordinary cusp: $(x^3y-x^4+2yx^2-y^2=0)$	V. $Y(4X^3-4Z)+4X^2-1=0$
VI. Quartic with a tac-node cusp: $[(y-x^2)^2+2xy(y-x^2)+y^3=0]$	VI. $ZY+\frac{1}{45}Z^6-\frac{1}{12}Z^4$ $-\frac{1}{3}XZ^3-X^2+\frac{13}{36}=0$
VII. Quartic having a triple-point with coincident tangents: $(y=x^2+x^4)$	VII. $Y=ZX^2-Z^5-Z^8$
VIII. Quartic having a triple-point with coincident tangents and a point of undulation: $(y=x^4)$	VIII. $Y=ZX^2-Z^5$

These then are the eight different types of algebraic translation-surfaces that can be generated in four different ways. The preceding developments show that these are all the types of algebraic surfaces.* In a subsequent paper I intend to discuss the types of translation-surfaces other than algebraic connected with a unicursal quartic.

U. S. NAVAL ACADEMY, *May* 21, 1906.

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* We have excluded here Cayly's ruled minimal surface mentioned above.

Group-Characters of Various Types of Linear Groups.[*]

BY HERBERT E. JORDAN.

INTRODUCTION.

In an article entitled Über Gruppencharaktere[†] Frobenius has determined the group-characters of the group of all binary linear fractional substitutions of determinant unity (when in their normal forms), the coefficients being taken modulo p, an odd prime. In the present paper the same method is applied to more general types of groups.

In Part I we consider the group $H \equiv SLH(2, p^n), p > 2$, of all binary linear homogeneous substitutions in the $GF[p^n]$, of determinant unity. By the aid of two theorems due to Frobenius on the relation between the characters of a group and those of one of its quotient-groups, we deduce as a corollary the characters of the group $F \equiv LF(2, p^n), p > 2$, of all binary linear fractional substitutions in the $GF[p^n]$ of determinant unity (when in their normal forms). We have also obtained these characters directly by the method applied to the group H; the chief points of difference in the treatment are stated in foot-notes. The results are a direct generalization of those obtained by Frobenius. In Part II we consider the group $H_1 \equiv SLH(2, p^n), p = 2$. This is identical with the group $LF(2, p^n), p = 2$. Part III deals with the group F_1 of all binary linear fractional substitutions in the $GF[p^n]$, $p > 2$, of determinant not zero. The group H is treated with considerable detail; the others briefly.

Frobenius[‡] has determined by another method the group-characters of the groups $SLH(2, 3)$, $(2, 5)$, and of the alternating group on six letters of order 360, which is isomorphic with the group $LF(2, 3^2)$ of determinant unity. Burnside[*] has obtained the group-characters of the binary linear homogeneous group in the $GF[2^3]$ of order 504. The results in this paper agree with those for the above special groups.

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[*] The abstract of the above paper appeared in the *Bulletin of the American Mathematical Society*, April, 1904. Just recently Schur has computed by different methods the characters of these same types of groups: Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, Crelle, Vol. 132, 1906–7, (Heft 2).

[†] Berliner Sitzungsberichte, 1896, pp. 985-1021.

[‡] Ueber die Composition der Charaktere einer Gruppe, Berliner Sitzungsberichte, 1899, pp. 330-339.

I.

The Binary Linear Homogeneous Group H in the $GF[p^n]$, $p>2$, of Determinant Unity.

§1.

The order of the group H is $h=p^n(p^n-1)$.[†] For the substitution

$$R:\quad \begin{aligned} x' &= \alpha x+\beta y,\\ y' &= \gamma x+\delta y, \end{aligned} \qquad \alpha\delta-\beta\gamma=1,$$

we use the notation[‡] $R=\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$.

We first reduce the substitutions of H to their canonical forms.[§] For this purpose we consider the characteristic equation

$$K^2-K(\alpha+\delta)+1=0, \tag{1}$$

of the substitution $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$. If the roots of this equation are distinct we get the canonical form A or B:

$$A:\quad \begin{pmatrix}\rho, & 0\\ 0, & \rho^{-1}\end{pmatrix}, \qquad \rho \text{ a mark} \neq 0 \text{ of the } GF[p^n],$$

$$B:\quad \begin{pmatrix}\sigma, & 0\\ 0, & \sigma^{-1}\end{pmatrix}, \qquad \sigma \text{ a root of } \sigma^{s+1}=1,$$

according as the equation (1) is reducible or irreducible in the $GF[p^n]$. If the roots of (1) are equal they must be $+1$ or -1. We obtain then (possibly by a transformation of determinant not unity) the canonical form

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[*] Proc. Lond. Math. Soc., Series 2, Vol. I—Part 2, p. 116.

[†] We shall throughout denote p^n by s, except in the notation $GF[p^n]$.

[‡] For the substitution R taken fractionally we use the notation $R=\left(\dfrac{\alpha,\ \beta}{\gamma,\ \delta}\right)$. To the two substitutions $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$ and $\begin{pmatrix}-\alpha, & -\beta\\ -\gamma, & -\delta\end{pmatrix}$ of H corresponds the one substitution $\left(\dfrac{\alpha,\ \beta}{\gamma,\ \delta}\right)$ of F. We have therefore a two-to-one correspondence between H and F.

[§] Dickson Linear Groups, §§ 214-216, 225.

$$C:\ \begin{pmatrix}\pm 1, & \pm 1\\ 0, & \pm 1\end{pmatrix}.$$

We define $\varkappa = \frac{1}{2}(\alpha + \delta)$ as the invariant* of the substitution $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$. If two substitutions of H have the same invariant, they have the same characteristic equation, and therefore the same canonical form. If two substitutions U and V of H have the same canonical form, there exists† a binary linear homogeneous substitution W belonging to the $GF[p^n]$ (but not necessarily of determinant unity) such that $U = W^{-1} V W$. Precisely as in §225 (Dickson, Linear Groups) we can prove that if U and V have the same canonical form A or B, there exists a substitution W_1 of H which transforms U into V; also that every substitution of H of invariant ± 1 (except $\begin{pmatrix}-1, & 0\\ 0, & -1\end{pmatrix}$ and the identity) is conjugate within H to one or other of the types

$$C_0:\ \begin{pmatrix}\pm 1, & \pm 1\\ 0, & \pm 1\end{pmatrix},$$

$$C_1:\ \begin{pmatrix}\pm 1, & \pm \mu\\ 0, & \pm 1\end{pmatrix},$$

where μ is a particular not-square in the $GF[p^n]$; and further that the two types C_0, C_1 are not conjugate within H. Hence we have the result:

Two substitutions of H having the same invariant (not ± 1) are conjugate within H.

A) Let ρ be a primitive root of the $GF[p^n]$. The substitution $R = \begin{pmatrix}\rho, & 0\\ 0, & \rho^{-1}\end{pmatrix}$ is of period $s-1$. To study the conjugacy of the substitutions $R^a = \begin{pmatrix}\rho^a, & 0\\ 0, & \rho^{-a}\end{pmatrix}$ we transform R^a by $U = \begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$, $\alpha\delta - \beta\gamma = 1$, and obtain

$$V = U^{-1} R^a U = \begin{pmatrix}\alpha\delta\rho^a - \beta\gamma\rho^{-a}, & -\alpha\beta(\rho^a - \rho^{-a})\\ \gamma\delta(\rho^a - \rho^{-a}), & -\beta\gamma\rho^a + \alpha\delta\rho^{-a}\end{pmatrix}.$$

In order that V shall be identical with R^a (*i. e.*, U commutative with R^a) it is necessary that either $\alpha\beta = \gamma\delta = 0$, or $\rho^a - \rho^{-a} = 0$. The first alternative leads to two cases:

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* In the case of F we define $\kappa = \pm\frac{1}{2}(\alpha + \delta)$ as the invariant of the substitution $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$.

† Dickson, Linear Groups, § 216.

$$1)\quad \text{if } \beta = \gamma = 0 \text{ then } V = \begin{pmatrix} \rho^a, & 0 \\ 0, & \rho^{-a} \end{pmatrix} = R^a, \quad U = \begin{pmatrix} \alpha, & 0 \\ 0, & \alpha^{-1} \end{pmatrix};$$

$$2)\quad \text{if } \alpha = \delta = 0 \text{ then } V = \begin{pmatrix} \rho^{-a}, & 0 \\ 0, & \rho^{-a} \end{pmatrix} = R^{-a}, \quad U = \begin{pmatrix} 0, & \beta \\ -\beta^{-1}, & 0 \end{pmatrix}.$$

If $R^a \neq R^{-a}$ we have $s - 1$ substitutions U commutative with R^a; and therefore R^a is one of $s(s^2 - 1) \div (s - 1) = s(s + 1)$ conjugate substitutions. If $R^a = R^{-a}$ then $\rho^a = \rho^{-a}$, which is the second alternative. According as $\rho^a = +1$ or -1, R^a is the identity or $R^{\frac{s-1}{2}} \equiv \begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}$; each of these substitutions is conjugate only to itself. With the exception of these the powers of R are conjugate in pairs, thus representing $\frac{1}{2}(s - 3)$ classes of conjugate substitutions, each class containing $s(s + 1)$ substitutions.*

B) The group H is holoedrically isomorphic with the group† $G_{2,p,n}$ of substitutions

$$U = \begin{pmatrix} A, & B \\ -\bar{B}, & \bar{A} \end{pmatrix}, \quad (A\bar{A} + B\bar{B} = 1),$$

where $\bar{A} \equiv A^s$ is the conjugate of A with respect to the $GF[p^n]$. If σ is a primitive root of the equation $\sigma^{s+1} = 1$, so that $\bar{\sigma} = \sigma^{-1}$, then the substitution $S = \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma^{-1} \end{pmatrix}$ is of period $s + 1$. As above we find that the powers of S, except $S^{\frac{s+1}{2}} = \begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}$ and S^{s+1}, which is the identity, are conjugate in pairs, viz., S^b with S^{-b}. There are $\frac{1}{2}(s - 1)$ classes represented by the powers of S, each containing $s(s - 1)$ substitutions.‡

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* In the case of the group F if $R^a = R^{-a}$ either $\rho^a = \rho^{-a}$, *i. e.*, R^a is the identity, or $\rho^a = -\rho^{-a}$, which is possible only if $s - 1$ is divisible by 4; $R^a \equiv R^{\frac{s-1}{4}}$ is commutative with $s - 1$ substitutions U, and is therefore one of $\frac{1}{2}s(s + 1)$ conjugate substitutions. If we define ε as $+1$ or -1 according as s has the form $4l + 1$ or $4l - 1$, where l is an integer, then we have $\frac{1}{4}(s - 2 + \varepsilon)$ classes represented by the powers of R, each containing $(ss + 1)$ substitutions except the class of period two, which contains $\frac{1}{2}s(s + 1)$ substitutions.

† Dickson, Linear Groups, p. 132.

‡ The substitution $S = \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma^{-1} \end{pmatrix}$ of F is of period $\frac{1}{2}(s + 1)$; the substitutions S^b (not the identity) are conjugate in pairs except when $\varepsilon = -1$, and then $S^{\frac{s+1}{4}}$ is conjugate only to itself and is of period two. We have $\frac{1}{4}(s - \varepsilon)$ classes, each containing $s(s - 1)$ substitutions, except the class of period two, which contains $\frac{1}{2}s(s - 1)$ substitutions.

The numbers $\pm a$ $(\pm b)$ taken mod. $s-1$ (mod. $s+1$) will be called indifferently the index of the class represented by R^a (S^b). We have defined $\varkappa = \frac{1}{2}(\alpha+\delta)$ as the invariant of the substitution $\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$. The substitutions* R^a (S^b) are characterized by the property that $\varkappa^2-1$ is a square (not-square) in the $GF[p^n]$.

C) Consider the substitution

$$T_\mu = \begin{pmatrix}1, & \mu \\ 0, & 1\end{pmatrix}, \ \mu \text{ a mark} \neq 0 \text{ of the } GF[p^n].$$

Transforming T_μ by $U = \begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$, $\alpha\delta-\beta\gamma=1$, we obtain

$$V = U^{-1}\,T_\mu\,U = \begin{pmatrix}1-\alpha\gamma\mu, & \alpha^2\mu \\ -\gamma^2\mu, & 1+\alpha\gamma\mu\end{pmatrix}.$$

T_μ is commutative only with those substitutions U for which $\gamma=0$, $\alpha=\pm 1$, in number $2s$; hence T_μ is one of $s(s^2-1)\div 2s = \frac{1}{2}(s^2-1)$ conjugate substitutions. We observe that the conjugate substitutions T_μ and V have the property that μ and $\alpha^2\mu$ $(\alpha \neq 0)$, or μ and $\gamma^2\mu$ in case $\alpha=0$, are both squares or both not-squares in the $GF[p^n]$. This condition can easily be proved to be sufficient for the conjugacy of T_μ and V; *i. e.*, a substitution $Q = \begin{pmatrix}\alpha', & \beta' \\ \gamma', & \delta'\end{pmatrix}$, $\alpha'\delta'-\beta'\gamma'=1$, of invariant unity is conjugate to T_μ if μ and β' $(\neq 0)$, or μ and $-\gamma'$ in case $\beta'=0$, are both squares or both not-squares in the $GF[p^n]$.

The substitutions $\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$ (except the identity) of invariant $+1$ will be said to belong to the class (μ) or to the class (ν) according as β $(\neq 0)$, or $-\gamma$ if $\beta=0$, is a square or a not-square in the $GF[p^n]$. Similarly the substitutions $\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$, except $\begin{pmatrix}-1, & 0 \\ 0, & -1\end{pmatrix}$, of invariant -1 will be said to belong to the class† (m) or to the class (n) according as β $(\neq 0)$, or $-\gamma$ if $\beta=0$, is a square or a not-square in the $GF[p^n]$.

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* The substitutions of period two have the invariant zero.

† To the two classes (μ) and (m) of H corresponds the one class (μ) of F; and similarly for (ν) and (n).

The substitutions[*] $\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix}$ and $\begin{pmatrix}-1, & 0\\ 0, & -1\end{pmatrix}$ will be denoted by (λ) and (l) respectively.

The total number[†] of classes of conjugate substitutions is $\frac{1}{2}(s-3)+\frac{1}{2}(s-1)+4+2=s+4$.

Since the powers of R and S were shown to be conjugate with their reciprocals, a substitution W and its reciprocal W^{-1} belong to the same class except when W belongs to one of the classes (μ), (ν), (m), (n).

§ 2.

We define ε_κ to be $+1$ or -1 according as κ^2-1 is a square or a not-square in the $GF[p^n]$, where κ denotes henceforth any invariant except ± 1. In place of ε_0 we write ε.[‡]

The class whose invariant is κ is denoted by (κ). This notation is unique. For, two conjugate substitutions have the same invariant; and it has been proved that if two substitutions of H have the same invariant (not ± 1) they are conjugate. Instead of κ we shall nearly always use $\alpha, \beta, \gamma, \ldots$, and we shall denote the indices of the classes (α), (β), $(\gamma), \ldots$ by $\pm a$, $\pm b$, $\pm c, \ldots$ respectively. These indices are taken mod. $s-1$ or mod. $s+1$ according as κ^2-1 is a square or a not-square in the $GF[p^n]$.

Denoting by h_θ the number of substitutions in a class (θ) we have[§]

$$h_\lambda=h_l=1,\ h_\mu=h_\nu=h_m=h_n=\tfrac{1}{2}(s^2-1),\ h_\kappa=s(s+\varepsilon_\kappa).$$

If $\varepsilon_\alpha=\varepsilon$, $\varepsilon_\beta=-\varepsilon$, the numbers[‖] of classes (α) and (β) are $\frac{1}{2}(s-2-\varepsilon)$ and $\frac{1}{2}(s-2+\varepsilon)$ respectively.

We define[¶] ζ_κ as $+1$ or -1 according as $-2(1-\kappa)$ is a square or a not-square in the $GF[p^n]$. Then $\zeta_a=\varepsilon_a(-1)^a$.

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[*] The one corresponding substitution $\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix}$ of F will be denoted by (λ).

[†] For F the total number of classes is $\frac{1}{2}(s+5)$.

[‡] For the definition of ε compare p. 390, foot-note*.

[§] For F we have $h_\lambda=1$, $h_\mu=h_\nu=\frac{1}{2}(s^2-1)$, $h_\kappa=s(s+\varepsilon_\kappa)$, $h_0=\frac{1}{2}s(s+\varepsilon)$. The class (0) requires to be distinguished from the other classes (κ) more frequently in the case of F than in the case of H.

[‖] For F the numbers of classes (a) and (β) are $\frac{1}{4}(s-\varepsilon)$ and $\frac{1}{4}(s-2+\varepsilon)$ respectively.

[¶] In the case of F we define η_κ as $+1$, -1, or 0 according as $-2(1+\kappa)$ and $-2(1-\kappa)$ are both squares, both not-squares, or one a square and the other a not-square, in the $GF[p^n]$. We also define $2\eta_1=\varepsilon$. If $\varepsilon_a=\varepsilon$ then $\eta_a=(-1)^a\varepsilon$; if $\varepsilon_\beta=-\varepsilon$ then $\eta_\beta=0$; further, $\eta_\kappa=\frac{1}{2}(1+\varepsilon\varepsilon_\kappa)\zeta_\kappa$.

§ 3.

Three (distinct or equal) classes (α), (β), (γ) are called *concordant* if between their invariants there exists the relation[*]

$$\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta\gamma = 1\,; \tag{1}$$

otherwise they are called *discordant.* If we write (1) in the form

$$(\alpha^2 - 1)(\beta^2 - 1) = (\alpha\beta - \gamma)^2$$

it follows that $\varepsilon_\alpha = \varepsilon_\beta$; similarly $\varepsilon_\beta = \varepsilon_\gamma$. Hence three concordant classes must all be represented by powers of R or all by powers of S. If α and β are given we find that the classes whose invariants are

$$\gamma = \alpha\beta + \sqrt{(\alpha^2 - 1)(\beta^2 - 1)}, \qquad \delta = \alpha\beta - \sqrt{(\alpha^2 - 1)(\beta^2 - 1)} \tag{2}$$

are concordant with (α) and (β). If $\beta = \alpha$ then $\gamma = 2\alpha^2 - 1$; and therefore we have $\varepsilon_\alpha = \varepsilon_{2\alpha^2} - 1$.

Let r denote ρ or σ according as $\varepsilon_\alpha = +1$ or -1. Substituting the values $2\alpha = r^a + r^{-a}$, etc., in (1), and factoring, we obtain

$$(r^{a+b+c} - 1)(r^{-a-b+c} - 1)(r^{-a+b-c} - 1)(r^{a-b-c} - 1) = 0.$$

Hence $a \pm b \pm c \equiv 0$ (mod. $s - 1$ or $s + 1$ according as $\varepsilon_\alpha = +1$ or -1). The indices of the two classes (γ) and (δ) concordant with (α) and (β) are therefore

$$c \equiv a + b, \quad d \equiv a - b \;(\text{mod.}\, s - 1,\ s + 1 \text{ respectively}).$$

§ 4.

Let Θ, Φ, Ψ represent the substitutions of any three distinct or equal classes (θ), (ϕ), (ψ) respectively, and let (θ'), (ϕ'), (ψ') denote the classes of the inverse substitutions Θ^{-1}, Φ^{-1}, Ψ^{-1} respectively. If Θ, Φ, Ψ run through all the substitutions of their respective classes, we denote[†] by $h_{\theta\phi\psi}$ the number of times we obtain the relation $\Theta\Phi\Psi = E$ (the identity), or $\Theta\Phi = \Psi^{-1}$. The subscripts θ, ϕ, ψ may be permuted in any manner.

To obtain $h_{\alpha\beta\gamma}$ we determine $\dfrac{h_{\alpha\beta\gamma}}{h_\alpha}$; we take a *particular* substitution of (α), compound it with all the substitutions of (β),

$$\begin{pmatrix} \alpha, & 1 \\ \alpha^2 - 1, & \alpha \end{pmatrix} \begin{pmatrix} \xi, & \eta \\ \zeta, & 2\beta - \xi \end{pmatrix} = \begin{pmatrix} \alpha\xi + \eta(\alpha^2 - 1), & \xi + \alpha\eta \\ \alpha\zeta + (2\beta - \xi)(\alpha^2 - 1), & \zeta + \alpha(2\beta - \xi) \end{pmatrix},$$

and determine how many of the resulting substitutions belong to the class (γ').

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[*] For F this relation takes the form $\alpha^2 + \beta^2 + \gamma^2 \pm 2\alpha\beta\gamma = 1$.

[†] Frobenius, Über Gruppencharaktere, 1896, pp. 987, 988.

In order that the resulting substitutions may be of class $(\gamma') \equiv (\gamma)$[*] and have the determinant unity, we must have

$$\alpha\xi + \eta(\alpha^2 - 1) + \zeta + \alpha(2\beta - \xi) = 2\gamma,$$
$$\xi(2\gamma - \xi) - \eta\zeta = 1.$$

The number of distinct sets of solutions ξ, η, ζ of these equations will give $\frac{h_{\alpha\beta\gamma}}{h_\alpha}$. Eliminating ζ we obtain

$$(\xi - \beta)^2 - (\alpha^2 - 1)\left(\eta + \frac{\alpha\beta - \gamma}{\alpha^2 - 1}\right)^2 = \frac{(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta - \gamma)^2}{\alpha^2 - 1}.$$

If (α), (β), (γ) are discordant the right-hand side is distinct from zero, and we obtain $s - \varepsilon_\alpha$ sets of solutions[†]. If (α), (β), (γ) are concordant[‡] the right-hand side is zero, and we obtain $s + \varepsilon_\alpha(s - 1)$ sets of solutions. Hence $h_{\alpha\beta\gamma} = h + \varepsilon_\alpha s^2 (s + \varepsilon_\alpha)$ or h according as (α), (β), (γ) are concordant or discordant.

If we denote the substitutions of (μ) and (ν) by P and Q respectively, then according as $\varepsilon = 1$ or -1 will P^{-1} belong to (μ) or (ν), and Q^{-1} to (ν) or (μ) Hence $h_{\lambda\mu\nu} = \frac{1}{2} h_\mu (1 - \varepsilon)$. Similarly $h_{\lambda mn} = \frac{1}{2} h_\mu (1 - \varepsilon)$.

The group H is self-conjugate under the group of all binary linear homogeneous substitutions of determinant $\neq 0$; by a substitution of determinant a not-square in the $GF[p^n]$ the class (μ) is transformed into (ν), and (ν) into (μ), and simultaneously (m) into (n) and (n) into (m). Hence the notations (μ) and (ν) are interchangeable; likewise (m) and (n); furthermore the interchange of (μ) and (ν) must be accompanied by the interchange of (m) and (n), and vice versa.

To determine $h_{\mu\mu\nu}$ we compute $h_\mu (h_{\mu\nu\mu} + h_{\mu\nu\nu} + h_{\mu\nu\lambda})$; we take a definite substitution of (μ), compound it with all the substitutions of (ν),

$$\begin{pmatrix} 0, & 1 \\ -1, & 2 \end{pmatrix} \begin{pmatrix} \xi, & \eta \\ \zeta, & 2 - \xi \end{pmatrix} = \begin{pmatrix} -\eta, & \xi + 2\eta \\ -2 + \xi, & \zeta + 4 - 2\xi \end{pmatrix},$$

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[*] See last paragraph of §1.

[†] Dickson, Linear Groups, p. 46.

[‡] In the case of F we have the equations

$$(\xi - \beta)^2 - (\alpha^2 - 1)\left(\eta + \frac{\alpha\beta \pm \gamma}{\alpha^2 - 1}\right)^2 = \frac{(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta \pm \gamma)^2}{\alpha^2 - 1}.$$

If (α), (β), (γ) are concordant then one of the relations $(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta \pm \gamma)^2 = 0$ holds and not the other. Suppose that $(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta + \gamma)^2 = 0$; then the equations with the upper and lower signs have $s + \varepsilon_\alpha (s - 1)$ and $s - \varepsilon_\alpha$ sets of solutions respectively; in all $2(s - \varepsilon_\alpha) + \varepsilon_\alpha s$ sets of solutions.

and find how many of the resulting substitutions belong to the classes (λ), (μ), (ν) collectively, *i. e.*, have the invariant $+1$. We have therefore the equations

$$\xi(2-\xi)-\eta\zeta=1,$$
$$-\eta+\zeta+4-2\xi=2.$$

Eliminating ξ we get $(\eta+\zeta)^2=0$, or $-\zeta=\eta$. We can take for η every not-square in the $GF[p^n]$, thus obtaining $\frac{1}{2}(s-1)$ sets of solutions. But $h_{\mu\nu\mu}=h_{\nu\mu\nu}=h_{\mu\nu\nu}$; hence $h_{\mu\mu\nu}=\frac{1}{4}h_\mu(s-2+\varepsilon)$.

Proceeding in this way we get the following results.*

$h_{\alpha\beta\gamma}=h+\varepsilon_a s^2(s+\varepsilon_a)$ or h according as (α), (β), (γ) are concordant or discordant,
$h_{aa\lambda}=h_a$, $h_{aal}=0$, $h_{aa\mu}=h_{aa\nu}=\frac{1}{2}h(1+\varepsilon_a)$, $h_{aam}=h_{aan}=\frac{1}{2}h$ if $\alpha \neq 0$,
$h_{oom}=h_{oon}=\frac{1}{2}h(1+\varepsilon)$, $h_{-aal}=h_a$, $h_{-aam}=\frac{1}{2}h(1+\varepsilon_a)$,
$h_{a\beta\lambda}=.0$, $h_{a\beta\mu}=h_{a\beta\nu}=h_{a\beta m}=h_{a\beta n}=\frac{1}{2}h$,
$h_{a\lambda l}=h_{a\lambda\mu}=h_{a\lambda\nu}=h_{a\lambda m}=h_{a\lambda n}=h_{al\mu}=h_{al\nu}=h_{alm}=h_{aln}=0$,
$h_{a\mu\mu}=h_{a\nu\nu}=h_{amm}=h_{ann}=\frac{1}{4}h(1+\varepsilon\zeta_a)$, $h_{a\mu\nu}=h_{amn}=\frac{1}{4}h(1-\varepsilon\zeta_a)$,
$h_{a\mu m}=h_{a\nu n}=\frac{1}{4}h(1+\varepsilon_a\zeta_a)$, $h_{am\nu}=h_{a\mu n}=\frac{1}{4}h(1-\varepsilon_a\zeta_a)$,
$h_{l\mu m}=h_{l\nu n}=h_{\lambda\mu\mu}=h_{\lambda\nu\nu}=h_{\lambda mm}=h_{\lambda nn}=\frac{1}{2}h_\mu(1+\varepsilon)$,
$h_{\lambda m\nu}=h_{\lambda\mu n}=h_{\lambda\mu m}=h_{\lambda\nu n}=h_{l\mu\mu}=h_{l\nu\nu}=h_{l\mu\nu}=h_{lmn}=0$,
$h_{lmm}=h_{lnn}=h_{\mu\mu n}=h_{m\nu\nu}=h_{\mu m\nu}=h_{\mu\nu n}=0$,
$h_{mmn}=h_{mnn}=0$, $h_{\lambda\mu\nu}=h_{\lambda mn}=h_{l\mu n}=h_{lm\nu}=\frac{1}{2}h_\mu(1-\varepsilon)$,
$h_{\mu\mu\mu}=h_{\nu\nu\nu}=h_{\mu mm}=h_{\nu nn}=\frac{1}{4}h_\mu(s-2-3\varepsilon)$,
$h_{\mu\mu\nu}=h_{\mu\nu\nu}=h_{\mu nn}=h_{mm\nu}=h_{\mu mn}=h_{m\nu n}=\frac{1}{4}h_\mu(s-2+\varepsilon)$,
$h_{\mu\mu m}=h_{\nu\nu n}=h_{mmm}=h_{nnn}=s\,h_\mu$.

§ 5.

The value of a group-character χ for any class (θ) is denoted by χ_θ; but sometimes the value of a group-character for a class represented by a power of R or S is denoted by $\chi(R^a)$ or $\chi(S^b)$ respectively. Also $\chi_\lambda=f$.

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* These results reduce to a very few in the case of F, since $(l)=(\lambda)$, $(m)=(\mu)$, $(n)=(\nu)$. For F we have the following additional results: $h_{oa\beta}=2h+\varepsilon s^2(s+\varepsilon)$ or $2h$ according as (o), (a), (β) are concordant or discordant; $h_{ooo}=\frac{1}{2}h$, $h_{ooa}=h$, $h_{oo\lambda}=\frac{1}{2}s(s+\varepsilon)$, $h_{oo\mu}=h_{oo\nu}=\frac{1}{2}h(1+\varepsilon)$, $h_{oa\mu}=h_{oa\nu}=h$, $h_{o\mu\mu}=h_{o\nu\nu}=\frac{1}{2}h(1+\varepsilon\eta)$, $h_{o\mu\nu}=\frac{1}{2}h(1-\varepsilon\eta)$, where $\eta=\eta_o$.

We make the following abbreviations:*

$$x = \tfrac{1}{2}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_\kappa \chi_\kappa,$$

$$y = \tfrac{\epsilon}{4}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_\kappa \zeta_\kappa \chi_\kappa,$$

$$z = \tfrac{\epsilon}{4}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_\kappa \epsilon_\kappa \zeta_\kappa \chi_\kappa.$$

From the relation †

$$h_\theta h_\psi \chi_\theta \chi_{\psi'} = f \sum_\phi h_{\theta'\phi\psi} \chi_\phi,$$

where (θ), (ϕ), (ψ) are any three classes, we derive the following set of equations ‡

$$\chi_l^2 = f^2, \tag{1}$$

$$\chi_a \chi_l = f \chi_{-a}, \tag{2}$$

$$s\chi_a \chi_\beta = fx, \quad (\epsilon_\beta = -\epsilon_a), \tag{3}$$

$$\frac{s(s+\epsilon_a)}{f} \chi_a \chi_\beta = x(s-\epsilon_a) + \epsilon_a s(\chi_\gamma + \chi_\delta), \quad (\epsilon_\beta = \epsilon_a,\ a \neq -\beta), \tag{4}$$

where γ and δ are determined by (3), §3,

$$\frac{s(s+\epsilon_a)}{f} \chi_a \chi_{-a} = \chi_l + x(s-\epsilon_a) + \tfrac{1}{2}\epsilon_a(s-\epsilon_a)(\chi_m + \chi_n) + \epsilon_a s \chi_{-(2a^2-1)}, \tag{5}$$

$$\frac{s(s+\epsilon_a)}{f} \chi_a^2 = f + x(s-\epsilon_a) + \tfrac{1}{2}\epsilon_a(s-\epsilon_a)(\chi_\mu + \chi_\nu) + \epsilon_a s \chi_{2a^2-1}, \tag{6}$$

$$\frac{s(s+\epsilon)}{f} \chi_0^2 = f + x(s-\epsilon) + \tfrac{1}{2}\epsilon(s-\epsilon)(\chi_\mu + \chi_\nu + \chi_m + \chi_n), \tag{7}$$

$$\frac{s+\epsilon_a}{f} \chi_a \chi_\mu = x + \epsilon_a \chi_a + \tfrac{1}{2}\zeta_a(\chi_\mu - \chi_\nu) + \tfrac{1}{2}\epsilon\epsilon_a \zeta_a(\chi_m - \chi_n), \tag{8}$$

$$\frac{s+\epsilon_a}{f} \chi_a \chi_\nu = x + \epsilon_a \chi_a - \tfrac{1}{2}\zeta_a(\chi_\mu - \chi_\nu) - \tfrac{1}{2}\epsilon\epsilon_a \zeta_a(\chi_m - \chi_n), \tag{8a}$$

$$\frac{s+\epsilon_a}{f} \chi_a \chi_m = x + \epsilon_a \chi_{-a} + \tfrac{1}{2}\epsilon\epsilon_a \zeta_a(\chi_\mu - \chi_\nu) + \tfrac{1}{2}\zeta_a(\chi_m - \chi_n), \tag{9}$$

$$\frac{s+\epsilon_a}{f} \chi_a \chi_n = x + \epsilon_a \chi_{-a} - \tfrac{1}{2}\epsilon\epsilon_a \zeta_a(\chi_\mu - \chi_\nu) - \tfrac{1}{2}\zeta_a(\chi_m - \chi_n), \tag{9a}$$

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[*] In the case of F we make the abbreviations

$$x = \chi_0 + \chi_\mu + \chi_\nu + 2\sum_\kappa \chi_\kappa, \qquad \kappa \neq 0,$$
$$y = \eta_0 \chi_0 + \eta_1(\chi_\mu + \chi_\nu) + 2\sum_\kappa \eta_\kappa \chi_\kappa.$$

[†] Über Gruppencharaktere, p. 994.

[‡] We get the equations for F from (3), (4), (6)—(8*a*), (10)—(10*b*) if we set $\chi_m = \chi_\mu$, $\chi_n = \chi_\nu$, and remember that $\tfrac{1}{2}(1 + \epsilon\epsilon_a)\zeta_a = \eta_a$.

$$\frac{s^2-1}{fs}\,\chi_\mu^2 = x+\varepsilon y+(1+\varepsilon)\frac{f}{s}-\frac{s+2\varepsilon+4}{4s}(\chi_\mu+\chi_\nu) - \frac{1}{s}(\chi_\mu-\chi_\nu)+\tfrac{1}{4}(\chi_m+\chi_n)+\varepsilon(\chi_m-\chi_n), \tag{10}$$

$$\frac{s^2-1}{fs}\,\chi_\nu^2 = x+\varepsilon y+(1+\varepsilon)\frac{f}{s}-\frac{s+2\varepsilon+4}{4s}(\chi_\mu+\chi_\nu)+\frac{1}{s}(\chi_\mu-\chi_\nu) +\tfrac{1}{4}(\chi_m+\chi_n)-\varepsilon(\chi_m-\chi_n), \tag{10a}$$

$$\frac{s^2-1}{fs}\,\chi_\mu\chi_\nu = x-\varepsilon y+(1-\varepsilon)\frac{f}{s}+\frac{s+2\varepsilon-4}{4s}(\chi_\mu+\chi_\nu)-\tfrac{1}{4}(\chi_m+\chi_n), \tag{10b}$$

$$\frac{s^2-1}{fs}\,\chi_\mu\chi_m = x+z+\frac{1+\varepsilon}{s}\chi_l+\frac{2-\varepsilon}{4}(\chi_\mu+\chi_\nu)+\varepsilon(\chi_\mu-\chi_\nu) -\frac{\varepsilon s+2\varepsilon+4}{4s}(\chi_m+\chi_n)-\frac{1}{s}(\chi_m-\chi_n), \tag{11}$$

$$\frac{s^2-1}{fs}\,\chi_\nu\chi_n = x+z+\frac{1+\varepsilon}{s}\chi_l+\frac{2-\varepsilon}{4}(\chi_\mu+\chi_\nu)-\varepsilon(\chi_\mu-\chi_\nu) -\frac{\varepsilon s+2\varepsilon+4}{4s}(\chi_m+\chi_n)+\frac{1}{s}(\chi_m-\chi_n), \tag{11a}$$

$$\frac{s^2-1}{fs}\,\chi_\mu\chi_n = \frac{s^2-1}{fs}\,\chi_m\,\chi_\nu = x-z+\frac{1-\varepsilon}{s}\chi_l -\frac{2-\varepsilon}{4}(\chi_\mu+\chi_\nu)+\frac{\varepsilon s+2\varepsilon-4}{4s}(\chi_m+\chi_n), \tag{11b}$$

$$\chi_l\chi_\mu = f\chi_m, \tag{12}$$

$$\chi_l\chi_\nu = f\chi_n. \tag{12a}$$

I. We seek first those solutions for which χ_μ and χ_ν are distinct. Then, according to (12) and (12a), χ_m and χ_n are distinct. From (1) we find $\chi_l=\pm f$.

Suppose first that $\chi_l=f$. Then $\chi_m=\chi_\mu$ and $\chi_n=\chi_\nu$, and from (8) and (8a) we obtain $\frac{s+\varepsilon\alpha}{f}\chi_\alpha=\zeta_\alpha(1+\varepsilon\varepsilon_\alpha)$. If $\varepsilon_\alpha=-\varepsilon$ then $\chi_\alpha=0$; if $\varepsilon_\alpha=\varepsilon$, and if we set the proportionality factor* $f=\tfrac{1}{2}(s+\varepsilon)$, we obtain $\chi_\alpha=\zeta_\alpha$. Hence $\chi(R^a)=\frac{(1+\varepsilon)(-1)^a}{2}$, $\chi(S^b)=-\frac{(1-\varepsilon)(-1)^b}{2}$. According to (3), $x=0$. If in (6) $\varepsilon_\alpha=-\varepsilon$, then $\chi_\mu+\chi_\nu=\varepsilon$, and $y=\tfrac{1}{2}+\sum_\kappa\zeta_\kappa^2$, where $\varepsilon_\kappa=\varepsilon$. The number of classes (κ) for which $\varepsilon_\kappa=\varepsilon$ is $\tfrac{1}{2}(s-2-\varepsilon)$, and therefore $y=\tfrac{1}{2}(s-1-\varepsilon)$. Similarly $z=\frac{\varepsilon}{2}(s-2)$. From (10b) we get $4\chi_\mu\chi_\nu=1-\varepsilon s$;

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[*] The proportionality factor may be chosen arbitrarily. See Über Gruppencharaktere, p. 999.

also we already have $\chi_\mu + \chi_\nu = \varepsilon$. Hence $\chi_\mu = \chi_m = \frac{1}{2}(\varepsilon \pm \sqrt{\varepsilon s})$, $\chi_\nu = \chi_n = \frac{1}{2}(\varepsilon \mp \sqrt{\varepsilon s})$. These values of the characters will be found to satisfy all the equations.

Let next $\chi_l = -f$. Then $\chi_m = -\chi_\mu$, $\chi_n = -\chi_\nu$. From (8) and (8a) we get as before $\frac{s+\varepsilon_a}{f}\chi_a = \zeta_a(1 - \varepsilon\varepsilon_a)$. If $\varepsilon_a = \varepsilon$, $\chi_a = 0$; if $\varepsilon_a = -\varepsilon$, and if we set $f = \frac{1}{2}(s - \varepsilon)$, we obtain* $\chi_a = \zeta_a$. According to (3), $x = 0$; if in (6) $\varepsilon_a = \varepsilon$ then $\chi_\mu + \chi_\nu = -\varepsilon$, and therefore $\chi_m + \chi_n = \varepsilon$. Also $y = \frac{1}{2}(s - 2 + \varepsilon)$ and $z = -\frac{1}{2}(s - 2 + \varepsilon)$. From (10b) we obtain $4\chi_\mu\chi_\nu = 1 - \varepsilon s$; this combined with $\chi_\mu + \chi_\nu = -\varepsilon$ gives $\chi_\mu = -\chi_m = \frac{1}{2}(-\varepsilon \pm \sqrt{\varepsilon s})$, $\chi_\nu = -\chi_n = \frac{1}{2}(-\varepsilon \mp \sqrt{\varepsilon s})$.

II. For all other solutions $\chi_\mu = \chi_\nu$, and therefore $\chi_m = \chi_n$. Instead of equations (10)–(11b) we shall use the following which are obtained from them by addition and subtraction:

$$\frac{s^2-1}{f}\chi_\mu^2 = sx + f - 2\chi_\mu, \tag{10'}$$

$$2\varepsilon s y = (s + 2\varepsilon)\chi_\mu - s\chi_m - 2\varepsilon f, \tag{10''}$$

$$\frac{s^2-1}{f}\chi_\mu\chi_m = sx + \chi_l - 2\chi_m, \tag{11'}$$

$$2\varepsilon s z = s(1 - 2\varepsilon)\chi_\mu + (s+2)\chi_m - 2\chi_l. \tag{11''}$$

We seek first those solutions for which x is distinct from zero. According to (3) none of the characters χ_a can be zero; and $\chi_a = \chi_\gamma$ if $\varepsilon_a = \varepsilon_\gamma$, *i. e.*, all characters χ_a are equal for which ε_a has the same sign. Since $\chi_a = \chi_{-a}$ it follows from (8) and (9) that $\chi_m = \chi_\mu$, and therefore from (12) that $\chi_l = f$.

Let $\varepsilon_a = \varepsilon$, $\varepsilon_\beta = -\varepsilon$. Then* $y = \varepsilon + \chi_a \sum\limits_a \zeta_a + \chi_\beta \sum\limits_\beta \zeta_\beta = \varepsilon - \varepsilon\chi_a$. From (6) and (8) we obtain

$$\left.\begin{aligned} \chi_a &= \tfrac{1}{s}\{(s - \varepsilon)\chi_\mu + \varepsilon f\}, \\ \chi_\beta &= \tfrac{1}{s}\{(s + \varepsilon)\chi_\mu - \varepsilon f\}. \end{aligned}\right\} \tag{A}$$

In the sum $x = \frac{1}{2}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum\limits_{a,\beta}(\chi_a + \chi_\beta)$, $\varepsilon_a = \varepsilon$, $\varepsilon_\beta = -\varepsilon$, the numbers of characters χ_a, χ_β are $\frac{1}{2}(s - 2 - \varepsilon)$, $\frac{1}{2}(s - 2 + \varepsilon)$ respectively. Hence we have

$$x = 2\chi_\mu + \tfrac{1}{2}(s - 2 - \varepsilon)\chi_a + \tfrac{1}{2}(s - 2 + \varepsilon)\chi_\beta. \tag{B}$$

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* From the definition of x we get $\zeta_\kappa = -\varepsilon$ or 0 according as $\varepsilon_\kappa = +\varepsilon$ or $-\varepsilon$. Hence we have in general $\sum\limits_\kappa \zeta_\kappa = -\varepsilon$.

Eliminating χ_α and χ_β from (A) and (B) we obtain

$$sx = (s^2+1)\chi_\mu - f.$$

Substituting this value of sx in (10′) we obtain

$$\chi_\mu(\chi_\mu - f) = 0.$$

If first $\chi_\mu = 0$, let $f = s$; then $x = y = -1$, $\chi_\alpha = \varepsilon$, $\chi_\beta = -\varepsilon$, in general $\chi_\kappa = \varepsilon_\kappa$, and therefore $\chi(R^a) = 1$, $\chi(S^b) = -1$. Also $\chi_l = s$.

If secondly $\chi_\mu = f$, let $f = 1$; then $\chi_l = \chi_\mu = \chi_m = \chi_\alpha = \chi_\beta = 1$. Also $x = s$, $y = 0$.

III. For all other solutions $x = 0$. Then (3) becomes $\chi_\alpha\chi_\beta = 0$ $\varepsilon_\beta = -\varepsilon_\alpha$. Not all the characters χ_κ can be zero. For, if they were, by giving to ε_κ in (6) the values 1 and -1 in turn we would have $\chi_\mu + \chi_\nu = 0$, and therefore $f = 0$, which is inadmissible.

According to (3) either all $\chi_\alpha = 0$ for which $\varepsilon_\alpha = 1$, or all for which $\varepsilon_\alpha = -1$. Suppose first that $\chi_\alpha = 0$ in case $\varepsilon_\alpha = -1$; and let $\chi_l = f = s + 1$. Then, since not all the characters χ_α for $\varepsilon_\alpha = 1$ can be zero, we obtain from (8) $\chi_\mu = 1$; and from (12) $\chi_m = 1$. If $\varepsilon_\alpha = \varepsilon_\beta = 1$, and therefore $\varepsilon_\gamma = \varepsilon_\delta = 1$, we obtain from (4), (5), (6)

$$\chi_\alpha\chi_\beta = \chi_\gamma + \chi_\delta,\ \chi_\alpha\chi_{-\alpha} = \chi_{-(2a^2-1)} + 2,\ \chi_\alpha^2 = \chi_{2a^2-1} + 2.$$

If we set $\chi_\alpha = \xi_a$, $\xi_0 = \xi_{\frac{s-1}{2}} = 2$, these equations can be combined into one:

$$\xi_a\xi_b = \xi_{a+b} + \xi_{a-b},$$

where a and b may be distinct or equal. Let r be a new unknown; if we set $\xi_1 = r + r^{-1}$ it follows from $\xi_1\xi_1 = \xi_2 + \xi_0$ that $\xi_2 = r^2 + r^{-2}$; then from $\xi_1\xi_2 = \xi_3 + \xi_1$ it follows that $\xi_3 = r^3 + r^{-3}$; in general $\xi_a = r^a + r^{-a}$. From $\xi_{\frac{s-1}{2}} = 2$ we get $r^{\frac{s-1}{2}} = 1$. We obtain then the solutions

$$\chi_l = f = s + 1,\quad \chi_\mu = \chi_\nu = \chi_m = \chi_n = 1,$$
$$\chi_\alpha = r^a + r^{-a} \text{ if } \varepsilon_\alpha = 1,\quad \chi_\beta = 0 \text{ if } \varepsilon_\beta = -1.$$

From (10″), (11″) we find $y = z = -1$. The above solutions satisfy the equation $x = 0$ except when $r = 1$, and the equations $y = z = -1$ except when $r = -1$; and r can be -1 only when $\varepsilon = 1$. If $\varepsilon = 1$ the equation $r^{\frac{s-1}{2}} = 1$ has $\dfrac{s-5}{2}$ solutions distinct from ± 1; if $\varepsilon = -1$ it has $\dfrac{s-3}{2}$ solutions distinct from 1; in general it has $\frac{1}{2}(s - 4 - \varepsilon)$ admissible solutions. Since r^a and r^{-a} give the same value for $r^a + r^{-a}$ these solutions go in pairs, giving $\frac{1}{4}(s - 4 - \varepsilon)$ characters.

We next let $\chi_l = -f = -(s+1)$. Then $\chi_m = -1$; and (4), (5), (6) become

$$\chi_\alpha \chi_\beta = \chi_\gamma + \chi_\delta, \quad \chi_\alpha \chi_{-\alpha} = \chi_{-(2\alpha^2-1)} - 2, \quad \chi_\alpha^2 = \chi_{2\alpha^2-1} + 2.$$

Setting $\chi_\alpha = \xi_a$, $\xi_0 = 2$, $\xi_{\frac{s-1}{2}} = -2$ we get $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$. Let $\xi_1 = r_1 + r_1^{-1}$; then as above we obtain $\xi_a = r_1^a + r_1^{-a}$, and also $r_1^{\frac{s-1}{2}} = -1$. We have the following solutions:

$$f = s+1, \quad \chi_l = -(s+1), \quad \chi_\mu = \chi_\nu = 1, \quad \chi_m = \chi_n = -1,$$
$$\chi_\alpha = r_1^a + r_1^{-a} \text{ if } \varepsilon_\alpha = 1, \quad \chi_\beta = 0 \text{ if } \varepsilon_\beta = -1.$$

Now $x = 0$, $y = z = \varepsilon - 1$. The above solutions satisfy $x = 0$, and also $y = z = \varepsilon - 1$ except when $r_1 = -1$, which can happen only when $\varepsilon = -1$. These solutions furnish $\frac{1}{4}(s - 2 + \varepsilon)$ characters.

IV. Suppose finally that $\chi_\alpha = 0$ in case $\varepsilon_\alpha = 1$. Let $f = s - 1$. Assuming first that $\chi_l = f$ we get $\chi_m = \chi_\mu = -1$. From (4), (5), (6) we have

$$\chi_\alpha \chi_\beta = -\chi_\gamma - \chi_\delta, \quad \chi_\alpha \chi_{-\alpha} = -\chi_{-(2\alpha^2-1)} + 2, \quad \chi_\alpha^2 = -\chi_{2\alpha^2-1} + 2.$$

Setting $\chi_\alpha = -\xi_a$, $\xi_0 = \xi_{\frac{s+1}{2}} = 2$, we obtain $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$. If $\xi_1 = t + t^{-1}$ then $\xi_b = t^b + t^{-b}$, and $t^{\frac{s+1}{2}} = 1$. We have then the following solutions:

$$\chi_l = f = s - 1, \quad \chi_\mu = \chi_\nu = \chi_m = \chi_n = -1,$$
$$\chi_\alpha = 0 \text{ if } \varepsilon_\alpha = 1, \quad \chi_\beta = -(t^b + t^{-b}) \text{ if } \varepsilon_\beta = -1.$$

The equation $x = 0$ is satisfied except when $t = 1$; and the equations $y = -1$, $z = 1 - 2\varepsilon$ are satisfied except when $t = -1$, which can happen only when $\varepsilon = -1$. These solutions furnish $\frac{1}{4}(s - 2 + \varepsilon)$ characters.

Assuming next that $\chi_l = -f$ we get $\chi_\mu = -1$, $\chi_m = 1$; also

$$\chi_\alpha \chi_\beta = -\chi_\gamma - \chi_\delta, \quad \chi_\alpha \chi_{-\alpha} = -\chi_{-(2\alpha^2-1)} - 2, \quad \chi_\alpha^2 = -\chi_{2\alpha^2-1} + 2.$$

Setting $\chi_\alpha = -\xi_a$, $\xi_0 = 2$, $\xi_{\frac{s+1}{2}} = -2$, we obtain $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$. If $\xi_1 = t_1 + t_1^{-1}$, then $\xi_b = t_1^b + t_1^{-b}$, and $t_1^{\frac{s+1}{2}} = 1$. We have the following solutions:

$$f = s - 1, \quad \chi_l = -(s-1), \quad \chi_\mu = \chi_\nu = -1, \quad \chi_m = \chi_n = 1,$$
$$\chi_\alpha = 0 \text{ if } \varepsilon_\alpha = 1, \quad \chi_\beta = -(t_1^b + t_1^{-b}) \text{ if } \varepsilon_\beta = -1.$$

We find that $x = 0$ is satisfied by all these solutions; and that $y = -(1+\varepsilon)$ and $z = 1 + \varepsilon$ are satisfied by all except $t_1 = -1$, which can happen only when $\varepsilon = 1$. These solutions furnish $\frac{1}{4}(s - \varepsilon)$ characters.

The total number of characters thus obtained is

$$4 + 2 + \tfrac{1}{4}(s - 4 - \varepsilon) + \tfrac{1}{4}(s - 2 + \varepsilon) + \tfrac{1}{4}(s - 2 + \varepsilon) + \tfrac{1}{4}(s - \varepsilon) = s + 4.$$

which is equal to the number of classes of conjugate substitutions.

Finally, we readily find that for all these $s+4$ characters the second proportionality factor e is equal to f, where e is defined by

$$\frac{hf}{e} \doteq \sum_{\theta} h_\theta \chi_\theta \chi_{\theta'}.$$

Below is given a table of the group-characters, N denoting the number of characters in the respective columns.

N	1	1	2	2	$\frac{s-\varepsilon-4}{4}$	$\frac{s+\varepsilon-2}{4}$	$\frac{s+\varepsilon-2}{4}$	$\frac{s-\varepsilon}{4}$
χ_λ	1	s	$\frac{s+\varepsilon}{2}$	$\frac{s-\varepsilon}{2}$	$s+1$	$s+1$	$s-1$	$s-1$
χ_ι	1	s	$\frac{s+\varepsilon}{2}$	$-\frac{s-\varepsilon}{2}$	$s+1$	$-(s+1)$	$s-1$	$-(s-1)$
χ_μ	1	0	$\frac{\varepsilon\pm\sqrt{\varepsilon s}}{2}$	$\frac{-\varepsilon\pm\sqrt{\varepsilon s}}{2}$	1	1	-1	-1
χ_ν	1	0	$\frac{\varepsilon\mp\sqrt{\varepsilon s}}{2}$	$\frac{-\varepsilon\mp\sqrt{\varepsilon s}}{2}$	1	1	-1	-1
χ_m	1	0	$\frac{\varepsilon\pm\sqrt{\varepsilon s}}{2}$	$\frac{\varepsilon\mp\sqrt{\varepsilon s}}{2}$	1	-1	-1	1
χ_n	1	0	$\frac{\varepsilon\mp\sqrt{\varepsilon s}}{2}$	$\frac{\varepsilon\pm\sqrt{\varepsilon s}}{2}$	1	-1	-1	1
$\chi(R^a)$	1	1	$\frac{(1+\varepsilon)(-1)^a}{2}$	$\frac{(1-\varepsilon)(-1)^a}{2}$	r^a+r^{-a}	$r_1^a+r_1^{-a}$	0	0
$\chi(S^b)$	1	-1	$-\frac{(1-\varepsilon)(-1)^b}{2}$	$-\frac{(1+\varepsilon)(-1)^b}{2}$	0	0	$-t^b-t^{-b}$	$-t_1^b-t_1^{-b}$

where r, r_1, t, t_1, are the roots (except ± 1) of the respective equations $r^{\frac{s-1}{2}}=1$, $r_1^{\frac{s-1}{2}}=-1$, $t^{\frac{s+1}{2}}=1$, $t_1^{\frac{s+1}{2}}=-1$.

§ 6.

By the use of the following theorems, due to Frobenius, we are able to deduce the group-characters of F from those of H.

If G is an invariant subgroup of the group H then every character of $\frac{H}{G}$ is also a character of H.[*]

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[*] Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, Berliner Sitzungsberichte, 1897, p. 995.

In order that a character of H may belong to the group $\frac{H}{G}$ *it is necessary and sufficient that it have the same value for all elements of G. Then it has also equal values for every two elements of H which are equivalent mod. G.**

In the present case the invariant subgroup G of H is composed of the substitutions $(l) = \left(\begin{smallmatrix}-1, & 0\\ 0, & -1\end{smallmatrix}\right)$ and (λ) the identity. Then those and only those characters of H for which $\chi_l = \chi_\lambda$ belong to the group $\frac{H}{G} = F$; and since every character of F belongs to H, we obtain in this way all the characters of F. The classes (μ) and (m), also (ν) and (n), are equivalent mod. G. Hence we can write down at once the table of characters for F.

N	1	1	2	$\frac{s-\varepsilon-4}{4}$	$\frac{s+\varepsilon-2}{4}$
χ_λ	1	s	$\frac{s+\varepsilon}{2}$	$s+1$	$s-1$
χ_μ	1	0	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	1	-1
χ_ν	1	0	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	1	-1
$\chi(R^a)$	1	1	$\frac{(1+\varepsilon)(-1)^a}{2}$	$r^a + r^{-a}$	0
$\chi(S^b)$	1	-1	$-\frac{(1-\varepsilon)(-1)^b}{2}$	0	$-t^b - t^{-b}$

where r and t are the roots (except ± 1) of the respective equations $r^{\frac{s-1}{2}} = 1$, $t^{\frac{s+1}{2}} = 1$.

II.

The Binary Linear Homogeneous Group H_1 in the $GF[2^n]$.

The order of H_1 is $h = 2^n(2^{2n} - 1)$, and the determinant of each substitution is unity. The group is holoedrically isomorphic with the group of all binary linear fractional substitutions in the $GF[2^n]$.

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*Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen, Berliner Sitzungsberichte, 1898, p. 610.

We define $x = \alpha + \delta$ as the invariant of the substitution $\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$.

The substitution $R = \begin{pmatrix}\rho, & 0 \\ 0, & \rho^{-1}\end{pmatrix}$, where ρ is a primitive root of the $GF[2^n]$, generates a cyclic group of order $s - 1$. R^a is conjugate to R^{-a} and is always distinct from it. Hence the powers of R represent $\frac{s-2}{2}$ classes, each containing $s(s+1)$ substitutions.

Let σ be a primitive root of $\sigma^{s+1} = 1$. The substitution $S = \begin{pmatrix}\sigma, & 0 \\ 0, & \sigma^{-1}\end{pmatrix}$ is of period $s + 1$. S^b is conjugate to S^{-b} and is distinct from it; thus the powers of S represent $\frac{s}{2}$ classes, each containing $s(s-1)$ substitutions.

The substitution $T = \begin{pmatrix}1, & 1 \\ 0, & 1\end{pmatrix}$ of period two and invariant zero is one of $s^2 - 1$ conjugate substitutions. We denote this class by (0) and the identity by (λ).

The total number of classes of conjugate substitutions is $s + 1$.

Below is given the table of group-characters.

N	1	1	$2^{n-1} - 1$	2^{n-1}
χ_λ	1	2^n	$2^n + 1$	$2^n - 1$
χ_0	1	0	1	-1
$\chi(R^a)$	1	1	$r^a + r^{-a}$	0
$\chi(S^b)$	1	-1	0	$-t^b - t^{-b}$

where r and t are the roots (except unity) of the respective equations $r^{s-1} = 1$, $t^{s+1} = 1$. As before $e = f$.

III.

The Binary Linear Fractional Group F_1 in the $GF[p^n]$, $p > 2$, of all Determinants not Zero.

The order of F_1 is $h = s(s^2 - 1)$. The substitutions will be supposed written in the normal form, *i. e.*, of determinant unity or a particular not-square in the $GF[p^n]$.

We shall denote the determinant $\alpha\delta - \beta\gamma$ of the substitution $V = \begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$ by τ, where $\tau = 1$, or ν a particular not-square; and we shall call $\pm \frac{1}{2}(\alpha + \delta)$ the invariant of V.

If two substitutions have not the same determinant they are not conjugate. If two substitutions (neither the identity) have the same determinant and the same invariant, they are conjugate under F_1.

By canonical form theory we find that all the substitutions of the group can be reduced to one or another of the following canonical forms:

A) $R=\begin{pmatrix}\rho, & 0\\ 0, & \rho^{-1}\tau\end{pmatrix}$, ρ a mark $\neq 0$ of the $GF[p^n]$;

B) $S=\begin{pmatrix}\sigma, & 0\\ 0, & \sigma^{-1}\tau\end{pmatrix}$, σ a mark $\neq 0$ of the $GF[p^{2n}]$;

C) $T=\begin{pmatrix}1, & 1\\ 0, & 1\end{pmatrix}$,

where σ satisfies a quadratic equation belonging to and irreducible in the $GF[p^n]$.

A) The substitution

$$R=\begin{pmatrix}\rho, & 0\\ 0, & \rho^{-1}\nu\end{pmatrix}$$

where $\rho^2\nu^{-1}$ is a primitive root of the $GF[p^n]$, is of period $s-1$. With the exception of $R^{\frac{s-1}{2}}$ which is conjugate only to itself, R^a is conjugate to R^{-a} and is distinct from it. We have therefore $\frac{s-1}{2}$ classes represented by the powers of R, each containing $s(s+1)$ substitutions, except $R^{\frac{s-1}{2}}$, the class represented by which contains $\frac{1}{2}s(s+1)$ substitutions.

B) The group of all binary linear fractional substitutions in the $GF[p^n]$ of determinant $\neq 0$ is holoedrically isomorphic with the group[*] of binary hyperorthogonal substitutions in the $GF[p^{2n}]$ of determinant a mark of the $GF[p^n]$ when taken fractionally, viz.,

$$U=\begin{pmatrix}A, & B\\ -\bar{B}, & \bar{A}\end{pmatrix} \qquad (A\bar{A}+B\bar{B}=\pi),$$

where $\bar{A}\equiv A^s$ is the conjugate of A with respect to the $GF[p^n]$, and π is a mark $\neq 0$ of the $GF[p^n]$.

Consider the substitution

$$S=\begin{pmatrix}\sigma, & 0\\ 0, & \sigma^{-1}\nu\end{pmatrix},$$

where σ is a primitive root of the equation $\sigma^{s+1}=\nu$. Since ν is an *arbitrary* not-square we may suppose that it is a primitive root of the $GF[p^n]$. Then σ is a primitive root of the $GF[p^{2n}]$, and consequently S is of period $s+1$. With the exception of $S^{\frac{s+1}{2}}$ which is conjugate only to itself, S^b is conjugate to S^{-b} and is distinct from it. We have therefore $\frac{s+1}{2}$ classes represented by the powers of S, each containing $s(s-1)$ substitutions, except $S^{\frac{s+1}{2}}$, the class represented by which contains $\frac{1}{2}s(s-1)$ substitutions.

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[*] Dickson, Linear Groups, §144, Cor.

The classes represented by the powers of $R(S)$ are characterized by the property that $x^2-\tau$ is a square (not-square) in the $GF[p^n]$, where $\tau = \nu$ or 1 according as the index is odd or even.

The substitution

$$T_\mu = \begin{pmatrix} 1, \mu \\ 0, 1 \end{pmatrix}, \ \mu \text{ a mark} \neq 0 \text{ of the } GF[p^n],$$

of invariant ± 1 and determinant unity, is one of s^2-1 conjugate substitutions forming a class (μ).

The total number of classes of conjugate substitutions is $s+2$.

Below is given the table of group-characters.

N	1	1	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$
χ_λ	1	1	s	s	$s+1$	$s-1$
χ_μ	1	1	0	0	1	-1
$\chi(R^{2a})$	1	1	1	1	$r^{2a}+r^{-2a}$	0
$\chi(S^{2b})$	1	1	-1	-1	0	$-t^{2b}-t^{-2b}$
$\chi(R^{2a+1})$	1	-1	1	-1	$r^{2a+1}+r^{-(2a+1)}$	0
$\chi(S^{2b+1})$	1	-1	-1	1	0	$-t^{2b+1}-t^{-(2b+1)}$

where r and t are the roots (except ± 1) of $r^{s-1}=1$ and $t^{s+1}=1$ respectively. As before $e=f$.

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